

Form factors of bound states in the XXZ chain

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Abstract

This work focuses on the calculation of the large-volume behaviour of form factors of local operators in the XXZ spin-1/2 chain taken between the ground state and an excited state containing bound states. The analysis is rigorous and builds on various fine properties of the string solutions to the Bethe equations and certain technical hypotheses. These technical hypotheses are satisfied for a generic excited state. The results obtained in this work pave the way for extracting, starting from the first principles, the large-distance and long-time asymptotic behaviour of the XXZ chain's two-point functions just as the so-called edge singularities of their Fourier transforms.

Introduction

Form factors of local operators constitute the elementary objects encoding the dynamics of a model. Although form factors cannot be computed in closed form for a general model, the state of the art is much more satisfactory in the case of quantum integrable models. First explicit computations of form factors in such models go back to the works [21, 23, 46]. There the form factor of various 1+1 dimensional integrable quantum field theories have been obtained by means of the bootstrap axioms. The analysis of form factors in lattice quantum integrable models has been pioneered by Jimbo and Miwa [20] on the example of the XXZ spin-1/2 chain at anisotropy greater than 1. All the results mentioned so far were obtained for massive models directly in the infinite volume, this owing to the existence of a mathematically well-posed description in such a setting. Indeed, Eigenstates of infinite volume massive integrable models can be labelled by using continuous rapidities and well-defined isolated variables. This property allows one to characterise the infinite volume form factors in terms of a sequence of densities in the continuous rapidities. The situation becomes much more involved in models having a massless spectrum as the very presence of massless modes renders such a description impossible, see *e.g.* [40]. For this reason, one can only achieve a meaningful description of the form factors in massless models by first keeping the volume finite and then extracting their large-volume asymptotic behaviour. However, obtaining finite volume expression for the form factors turned out to be a rather challenging task, even in the case of quantum integrable models. In the case of interacting models, it could only be achieved recently thanks to the

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invention of the algebraic Bethe Ansatz [14] followed by the calculation of norms [35] and scalar products [44] of Bethe vectors and, finally, the resolution [30] of the quantum inverse scattering problem. All these ingredients put together led [30] to determinant based representations for the form factors of local operators of the finite volume XXZ spin-1/2 chain. In the massless antiferromagnetic phase of the model, the size of the determinants describing the ground to excited states form factors grows linearly with the model's volume. Thus, the analysis of the large-volume behaviour of the form factors demands to extract the large-size behaviour of the underlying class of determinants. Such a problem was first investigated by N. Slavnov in [45] who studied the large-size asymptotics of determinants describing the form factors of the current operator in the non-linear Schrödinger model. This analysis was then improved and extended in the works [25, 27]. There, the authors considered the massless regime of the XXZ chain at finite magnetic field and extracted the large-volume behaviour of the form factors taken between the ground state and a class of excited states of particle-hole type. This explicit control on the large-volume behaviour of the particle-hole form factors allowed to derive, on the level of heuristic but physically quite plausible arguments, the large-distance asymptotic behaviour of two [26] and multi-point [29] correlation functions in the XXZ chain. Furthermore, such a large-volume asymptotic behaviour permitted to identify the amplitudes arising in the long-distance asymptotic expansion as the thermodynamic limit of properly normalised in the volume form factors of local operators [24, 25]. It also led to establishing [40] a correspondence between a weak limit of operators in the lattice XXZ chain and certain vertex operators arising in the free boson model.

In fact, for massless models having a pure particle-hole spectrum such as the non-linear Schrödinger model, the control on the large-volume asymptotic behaviour of form factors was enough so as to derive [28] and even prove [39], under technical hypotheses of convergence of auxiliary series, the asymptotic behaviour of dynamical correlation functions in this model. The work [28] also confirmed the predictions for the value of the so-called edge exponents that were predicted earlier on by means of the non-linear Luttinger liquid approach [15, 16, 41].

The above stresses the prominent role that the large-volume behaviour of form factors in massless models plays in the analysis of various properties of a model's correlation functions. However, the study of dynamical properties of more complex models such as the spin-1/2 XXZ chain demands more work. Indeed, on top of particle-hole excitations, this model also exhibits bound states. These were argued, on many instances, see *e.g.* [1, 2, 33], to play a role in the dynamical properties of the chain. As a consequence, in order to deal appropriately with the time and space dependent properties of the chain one first needs to access to the large-volume asymptotic behaviour of the form factors of local operators involving, on top of particle-hole excitations, the bound states as well.

One should stress that more effort is needed to extend the techniques of the large-volume analysis of form factors to the case of form factors involving bound states. This stems, in particular, from the way the bound states are characterised within the Bethe Ansatz; these are described in terms of certain complex valued solutions to the Bethe equations which agglomerate, in the thermodynamic limit, into complexes called strings. A given string can be characterised in terms of its length and its central root, the so-called string centre. The first investigation of the large-volume L behaviour of quantities building up the XXZ chain's form factors between Eigenstates containing bound states concerned the norm of a Bethe Eigenvector. The authors of [22] took a formal large- L limit of the determinant representing the norm of a Bethe Eigenvector containing bound states and managed to reduce the dependence on the parameters of the bound state to solely one on the string centres. More recently, the authors of [7] generalised the former result to the *per se* case of form factors involving bound states. The rewriting of the form factor determinants they obtained was enough so as to allow them a numerical calculation of the form factors. However, the mentioned handlings do not appear to allow for a rigorous analysis, mainly due to the

difficulties associated with controlling the remainders within such an approach. In fact, owing to the exponential in the volume divergencies that appear in the entries of the form factor determinants due to the presence of bound states, the rigorous analysis of the large-volume behaviour of such form factors demands to have a very precise characterisation of the large- L behaviour of string solutions. The first full analysis of the large-volume behaviour of form factors involving bound states has been carried out in [13] for the massive regime of the XXZ chain. The analysis relied on certain technical hypotheses and on the results of [4, 10, 50] where higher level Bethe equations characterising the complex solutions to the Bethe equations above the ground state in this sector have been obtained. In fact, due to the importance played by the bound states in the XXZ chain, starting from the pioneering work of Bethe [5], a rather extensive literature has been devoted to the analysis of complex solutions to the Bethe equations. Until recently, a rigorous description of bound states could only be achieved for the ferromagnetic regime of the XXZ chain [3, 49]. In [38], the author managed to characterise, on rigorous grounds, the structure of a wide class of complex solutions in the massless regime of the antiferromagnetic chain in the presence of a finite magnetic field. I refer to that paper for a review of the history of complex solutions to the XXZ chain Bethe Ansatz equations. The results of [38] opened the way to a rigorous analysis of the large- L behaviour of bound state form factors in the XXZ chain that is developed in this work.

The goal of the present paper is to determine, on rigorous grounds, the large-volume asymptotic behaviour of the form factors of local operators in the XXZ spin-1/2 chain, this while keeping a uniform in respect to the excited states, control on the remainder. These large-volume asymptotics will then be used, in a subsequent publication, so as to demonstrate, under mild assumptions, the non-linear Luttinger liquid model-based predictions for the edge exponents characterising the singular behaviour of Fourier transforms of two-point functions in the chain and also extract the long-time and large-distance asymptotic behaviour of two-point functions in the model.

The paper is organised as follows. Section 1 introduces the model and states, without giving too much details on the building blocks, the overall form taken by the large- L asymptotic behaviour of form factors of local operators that is obtained in the present work. This section also contains a list of various notations that will be employed in the core of the paper. Section 2 is of technical nature and discusses all the solutions to the linear integral equations that drive the thermodynamics of the chain. Section 3 gathers various auxiliary results that are necessary for the analysis of the large-volume behaviour of form factors. Section 4 presents the starting, determinant-based, expressions for the form factors of local operators in the model. It also contains the precise statement, with all building blocks given explicitly, of the main theorem proven in this work. Sections 5, 6 and 7 focus on the extraction of the large-volume behaviour of the various sub-constituents of the form factors. The paper contains various appendices where several technical results of use to the analysis are established. Appendix A is devoted to the asymptotic analysis of the integral transforms that arise in the course of the analysis. Appendix B establishes a certain amount of bounds that appear helpful for the analysis developed in the core of the paper. Finally, Appendix C list certain identities involving the special functions that are used in the paper as well as an evaluation of certain auxiliary integrals that appear in the course of the analysis.

1 The model and main results

1.1 The model

The XXZ spin-1/2 chain refers to a system of interacting spins in one dimension described by the Hamiltonian

$$H_\Delta = J \sum_{a=1}^L \left\{ \sigma_a^x \sigma_{a+1}^x + \sigma_a^y \sigma_{a+1}^y + \Delta \sigma_a^z \sigma_{a+1}^z \right\}. \quad (1.1)$$

H_Δ is an operator on the Hilbert space $\mathfrak{h}_{XXZ} = \otimes_{a=1}^L \mathfrak{h}_a$ with $\mathfrak{h}_a \simeq \mathbb{C}^2$. The matrices σ^w , $w = x, y, z$ are the Pauli matrices and σ_a^w stands for the operator on \mathfrak{h}_{XXZ} which acts as the Pauli matrix σ^w on \mathfrak{h}_a and as the identity on all the other spaces appearing in the tensor product defining \mathfrak{h}_{XXZ} . The Hamiltonian depends on two coupling constants: $J > 0$ which represents the so-called exchange interaction and Δ which parametrises the anisotropy in the coupling between the spins in the longitudinal and transverse directions. In this paper I shall focus on the range of anisotropy $-1 < \Delta < 1$ which corresponds to the so-called massless anti-ferromagnetic regime. In the following, I shall adopt the parametrisation

$$\Delta = \cos(\zeta) \quad \text{with} \quad \zeta \in]0; \pi[. \quad (1.2)$$

The XXZ Hamiltonian commutes with the total spin operator $S^z = \sum_{a=1}^L \sigma_a^z$. It can thus be diagonalised in each sub-space $\mathfrak{h}_{XXZ}^{(N)}$ corresponding to a fixed Eigenvalue of S^z :

$$\mathfrak{h}_{XXZ}^{(N)} = \left\{ |v\rangle \in \mathfrak{h}_{XXZ} : S^z |v\rangle = (L - 2N) |v\rangle \right\} \quad \text{so that} \quad \mathfrak{h}_{XXZ} = \bigoplus_{N=0}^L \mathfrak{h}_{XXZ}^{(N)}. \quad (1.3)$$

As a consequence, one can embed H_Δ in an external longitudinal magnetic field h and rather focus on the Hamiltonian $H_{\Delta,h} = H_\Delta - hS^z/2$, this without altering the diagonalisation problem. The effect of the magnetic field will be to change the value of the integer N labelling the subspace $\mathfrak{h}_{XXZ}^{(N)}$ which contains the model's ground state.

When $-1 < \Delta < 1$ and for magnetic fields $h \geq h_c = 8J \cos^2(\zeta/2)$, the Hamiltonian $H_{\Delta,h}$ is in its ferromagnetic regime and the ground state belongs to the $\mathfrak{h}_{XXZ}^{(0)}$ subspace. When the magnetic field is below the critical value h_c , $0 \leq h < h_c$, the model is an antiferromagnet. Then, the ground state belongs to the subspace $\mathfrak{h}_{XXZ}^{(N)}$ with N such that $N/L \rightarrow D \in]0; 1/2[$. As will be discussed later on, the value of D is fixed by h . In the following, I will focus on the regime $0 < h < h_c$. The reason is that, on the one hand, when $h = 0$ and $L \rightarrow +\infty$, the structure of the complex solutions to the Bethe equations has been advocated [4, 10, 53] to change drastically in respect to the string picture argued by Bethe. A rigorous description of the $L \rightarrow +\infty$ behaviour of the complex solutions when $h = 0$ is still an open problem. Furthermore, owing to the unboundedness in L of the solutions to the ground state Bethe equations, *c.f.* [37], various brand new features will appear in the $L \rightarrow +\infty$ analysis of the form factors. On the other hand, for $h \geq h_c$, the problem is trivial.

1.2 The main result

As observed by Bethe [5] for $\Delta = 1$ and generalised to any Δ by Orbach [43], Eigenstates $|\Upsilon\rangle$ of $H_{\Delta,h}$ can be constructed by the so-called Bethe Ansatz. Within this approach, the Eigenstates $|\Upsilon\rangle$ are constructed as certain combinatorial sums depending on a set of auxiliary parameters $\Upsilon = \{\mu_a\}_1^{|\Upsilon|}$ which solve a system of transcendental equations, the so-called Bethe equations. The completeness of the Bethe Eigenstates

for the XXZ chain, as built from solutions to the Bethe equations, is a complicated issue which can however be settled for a certain inhomogeneous deformation of the model [48] or by adopting a slightly more general point of view, as it has been done for the XXX Heiseiberg chain [42]. I will not dwell on such issues in the present work, and simply consider Eigenstates $|\Upsilon\rangle$ which can be build from the Bethe Ansatz.

In the following, I will consider Eigenstates $|\Upsilon\rangle$ which have, in the thermodynamic limit, a finite excitation energy above the ground state. In this limit, the Eigenstate can be described in terms of

- massless excitations characterised by the integers $\{p_a^\nu\}_{a=1}^{n_\nu^{(p)}} \cup \{h_a^\nu\}_{a=1}^{n_\nu^{(h)}}$ with $\nu \in \{L, R\}$;
- massive excitations characterised by the parameters $\mathfrak{C} = \{\{c_p^{(a)}\}_{a=1}^{n_p^{(z)}}\}_{p=1}^{p_{\max}}$ and $\Upsilon_{\text{off}}^{(h)} = \{\mu_a^{(h)}\}_{a=1}^{n_{\text{off}}^{(h)}}$
- Umklapp integers ℓ_ν and centred Umklapp integers $\ell_\nu^\mathbb{Z}$.

All these quantities are defined precisely in the core of the paper, *c.f.* Sections 3.4-3.5 and Section 3.9.

The form factors of local operators correspond to the expectation values of the local operators σ_1^γ taken between two Eigenstates of $H_{\Delta, h}$

$$\mathcal{F}_{\Upsilon; \Omega}^{(\gamma)} = \frac{\langle \Upsilon | \sigma_1^\gamma | \Omega \rangle}{\|\Upsilon\| \cdot \|\Omega\|}. \quad (1.4)$$

The main result of this paper is summarised in the theorem below.

Theorem 1.1. *Let Λ be the set of Bethe roots describing the ground state at magnetic field $0 < h < h_c$ and Υ be the set of Bethe roots described above and such that the roots of Υ are spaced at least as described in Hypothesis 3.3.*

Assume that the integers p_a^ν and h_a^ν and the parameters forming $\mathfrak{C} \cup \Upsilon_{\text{off}}^{(h)}$ are finite in number. Assume that the integers p_a^ν and h_a^ν and the parameters forming $\mathfrak{C} \cup \Upsilon_{\text{off}}^{(h)}$ are bounded in L . Given $\gamma = z$ or $\gamma = +$, then there exists $0 < r < 1/4$ such that the below asymptotic expansion holds

$$\begin{aligned} |\mathcal{F}_{\Upsilon; \Lambda}^{(\gamma)}|^2 &= \oint_{\partial \mathcal{D}_{0, r}} \prod_{\nu \in \{L, R\}} \left\{ \frac{G^2(1 - \sigma_\nu \mathfrak{F}_\nu^{(\varrho)})}{G^2(1 - \sigma_\nu [\mathfrak{F}_\nu^{(\varrho)} - \sigma_\nu \ell_\nu])} \frac{\mathcal{R}_{n_p^\nu, n_h^\nu}(\{p_a^\nu\}; \{h_a^\nu\} | -\sigma_\nu \mathfrak{F}_\nu^{(\varrho)})}{(L/2\pi)^{(\mathfrak{F}_\nu^{(\varrho)} - \sigma_\nu \ell_\nu)^2}} \right\} \\ &\times \frac{\mathcal{F}^{(\gamma)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_\nu^\mathbb{Z}\} | \varrho) \cdot (1 + O(\ln L/L))}{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \{L p_1'(\mu)\} \cdot \prod_{r=1}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \{L p_r'(c_a^{(r)})\}} \cdot \frac{d\varrho}{2i\pi\varrho}}. \quad (1.5) \end{aligned}$$

The structure of the answer is the following. The contour integral over the auxiliary variable ϱ plays the role of a regularisation.

The leading asymptotics can be split in two contributions. The one appearing on the first line of (1.5) corresponds to the massless modes. The pre-factor containing the Barnes- G functions is a normalisation constant. The function $\mathcal{R}_{n_p^\nu, n_h^\nu}$ should be thought of as the form factor density squared associated with the massless modes of the model. The contribution is weighted by the non-integer power of the volume $L^{-(\mathfrak{F}_\nu^{(\varrho)} - \sigma_\nu \ell_\nu)^2}$ which should be thought of as the fundamental spectral "volume" carried by the massless excitation. Here $\sigma_L = -1$ and $\sigma_R = 1$, $\ell_\nu^\mathbb{Z} \in \mathbb{Z}$ are the Umklapp integers associated with the excited state Υ and $\mathfrak{F}_{L/R}^{(\varrho)}$ is related to the value taken by the shift function associated with the excited state Υ on the left/right end of the Fermi zone.

The contribution appearing in the second line of (1.5) corresponds to the massive modes. The function $\mathcal{F}^{(\gamma)}$ should be thought of as the form factor density squared associated with the massive modes of the model. $\mathcal{F}^{(\gamma)}$ is a smooth function of the variables $\Upsilon_{\text{off}}^{(h)}$ and \mathfrak{C} . The contribution is weighted by an integer power of the volume: exactly one power for each massive parameter present in $\mathfrak{C} \cup \Upsilon_{\text{off}}^{(h)}$. The functions $p'_r/2\pi$, represent the density at which each of the variables condense in the thermodynamic limit.

The expressions for the various constituents of the leading asymptotics are a bit bulky and their definition demands a certain amount of auxiliary objects. Such details are thus postponed to Section 4.2 where, also, a slightly more general version of the result is presented in Theorem 4.2.

1.3 Main notations

- Given a set S , $|S|$ stands for its cardinal, $\mathbf{1}_S$ for its indicator function.
- I_α stands for the interval $I_\alpha = [-\alpha; \alpha]$.
- Given a set S , $\mathcal{S}_\delta(S)$ stands for a δ -neighbourhood of S , namely $\mathcal{S}_\delta(S) = \{z \in \mathbb{C} : |d(z, S)| < \delta\}$ where $d(z, S)$ is the distance of the point z to the set S that is induced by the canonic distance on \mathbb{C} . For instance, $\mathcal{S}_\delta(\mathbb{R})$ is the strip of width 2δ centred on \mathbb{R} .
- $\mathcal{D}_{z,\delta}$ stands for the open disk of radius δ centred at z .
- Given two sets A, B , I adopt the shorthand notation for products and sums involving

$$\prod_{\substack{\lambda, \mu \in \\ A \setminus B}} f(\lambda) \equiv \frac{\prod_{\lambda \in A} f(\lambda)}{\prod_{\lambda \in B} f(\lambda)} \quad \text{and} \quad \sum_{\substack{\lambda, \mu \in \\ A \setminus B}} f(\lambda) \equiv \sum_{\lambda \in A} f(\lambda) - \sum_{\lambda \in B} f(\lambda). \quad (1.6)$$

In the case when $B \subset A$, this convention reproduces the value of a product or sum over elements of the set $A \setminus B$.

- The function \ln refers to the principal branch of the logarithm. Unless stated otherwise, it is this branch that will be used in the formulae.
- Given $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x , namely the closest integer lower or equal to x .
- Given a closed non-intersecting curve \mathcal{C} in $\mathbb{C} \setminus \{i\pi\mathbb{Z}\}$, $\text{Ext}(\mathcal{C})$ and $\text{Int}(\mathcal{C})$ denote, respectively, the interior and exterior of \mathcal{C} in $\mathbb{C} \setminus \{i\pi\mathbb{Z}\}$.
- Given an open subset $O \subset \mathbb{C}$, ∂O denotes its canonically oriented boundary.
- In general $C, c, \widetilde{C}, C', \dots$ denote positive constants appearing in the various bounds. The value of these constants may change from one line of an equation to another without further specifications.
- $W_k^\infty(\mathbb{R})$ stands for the L^∞ -based Sobolev space of order $k \in \mathbb{N}$, namely

$$W_k^\infty(\mathbb{R}) = \left\{ f : f^{(p)} \in L^\infty(\mathbb{R}) \quad 0 \leq p \leq k \right\} \quad \text{with norm} \quad \|f\|_{W_k^\infty(\mathbb{R})} = \max_{0 \leq p \leq k} \|f^{(p)}\|_{L^\infty(\mathbb{R})}. \quad (1.7)$$

- Let Γ and G be the Euler Gamma and the Barnes- G functions. I adopt the hypergeometric-like notation for products and ratios of such functions

$$\Gamma\left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_\ell \end{matrix}\right) = \frac{\prod_{s=1}^k \Gamma(a_s)}{\prod_{s=1}^\ell \Gamma(b_s)} \quad \text{and} \quad G\left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_\ell \end{matrix}\right) = \frac{\prod_{s=1}^k G(a_s)}{\prod_{s=1}^\ell G(b_s)}. \quad (1.8)$$

2 The special functions at play

The observables associated with the thermodynamic limit of integrable models are described through a collection of solutions to linear integral equations. This section provides the description of all the functions of this type that arise in the context of the present analysis. I first discuss the bare quantities which appear as driving terms in the linear integral equations and then introduce the solutions to these equation, the so-called dressed quantities.

2.1 The bare quantities

2.1.1 Integral kernels

The function

$$K(\lambda | \eta) = \frac{1}{2i\pi} \left\{ \coth(\lambda - i\eta) - \coth(\lambda + i\eta) \right\} = \frac{\sin(2\eta)}{2\pi \sinh(\lambda + i\eta) \sinh(\lambda - i\eta)} \quad (2.1)$$

plays an important role in the analysis. When η is specialised to ζ introduced in (1.2), $K(\lambda | \zeta)$ corresponds to the integral kernel of the operator that drives the linear integral equations describing the thermodynamic observables in the model. In fact, below, whenever the auxiliary argument will be dropped, $K(\lambda)$ will stand precisely for this function:

$$K(\lambda) \equiv K(\lambda | \zeta). \quad (2.2)$$

Moreover, sums of $i\zeta$ -shifted kernels K arise in the description of the quantities associated with the bound states:

$$K_{r,s}(\omega) \equiv \sum_{\ell=1}^r \sum_{k=1}^s K\left(\omega + i\frac{\zeta}{2}(r - s - 2(\ell - k))\right). \quad (2.3)$$

These sums are symmetric in r, s and can be recast in terms of a reduced number of functions $K(\lambda | \eta)$ evaluated at different values of η as

$$\begin{aligned} K_{r,s}(\omega) = & K\left(\omega \mid \frac{1}{2}\zeta(r + s)\right) + (2 - \delta_{1,s} - \delta_{1,r}) \cdot K\left(\omega \mid \frac{1}{2}\zeta(r + s - 2)\right) \\ & + \sum_{p=\lceil \frac{r-s+2}{2} \rceil}^{r-2} (w_{p-1}^{(r,s)} - w_{p+1}^{(r,s)}) \cdot K\left(\omega \mid \frac{1}{2}\zeta(r - s - 2p)\right) \end{aligned} \quad (2.4)$$

where $\delta_{a,b}$ is the Kronecker symbol and

$$w_p^{(r,s)} \equiv \min(r, s + p) - \max(0, p) = w_{r-s-p}^{(r,s)}. \quad (2.5)$$

Note that, as a particular case, one has

$$K_{1,r}(\omega) = K\left(\omega \mid \frac{1}{2}\zeta(r+1)\right) + (1 - \delta_{1,r}) \cdot K\left(\omega \mid \frac{1}{2}\zeta(r-1)\right). \quad (2.6)$$

One has analogous identities to (2.4) relatively to the double product

$$\Phi_{r,p}(\lambda) = \Phi_{p,r}(\lambda) = \prod_{k=1}^p \prod_{s=1}^r \left\{ \frac{\sinh\left(\lambda + \frac{i}{2}\zeta[p-r-2(k-s)]\right)}{\sinh\left(\lambda + \frac{i}{2}\zeta[p-r-2(k-s+1)]\right)} \right\}. \quad (2.7)$$

For general r and p , it can be recast as

$$\Phi_{r,p}(\lambda) = \prod_{\ell=\lceil \frac{r-p+1}{2} \rceil}^{r-1} \left\{ \frac{\sinh\left(\lambda + i\zeta\left[\frac{p-r}{2} + \ell\right]\right)}{\sinh\left(\lambda - i\zeta\left[\frac{p-r}{2} + \ell + 1\right]\right)} \right\}^{w_\ell^{(r,p)} - w_{\ell+1}^{(r,p)}}. \quad (2.8)$$

This expression slightly simplifies when one of the integers is set to one:

$$\Phi_{1,p}(\lambda) = \Phi_{p,1}(\lambda) = \frac{\sinh\left(\lambda + i\zeta\frac{p-1}{2}\right)}{\sinh\left(\lambda - i\zeta\frac{p+1}{2}\right)}. \quad (2.9)$$

The representation (2.8) allows one to deduce the structure of the zeroes and poles of $\Phi_{r,p}$ close to \mathbb{R} .

Let $0 \leq \ell \leq p_{\max}$. Then there exists $\delta > 0$ such that

- when $r \neq \ell$ $\Phi_{r,\ell}$ has no poles or zeroes in $\mathcal{S}_\delta(\mathbb{R})$;
- $\Phi_{r,r}$ has no poles in $\mathcal{S}_\delta(\mathbb{R})$ and a unique zero at 0 which has multiplicity one.

2.1.2 The bare phases

The bare phase $\vartheta(\lambda \mid \eta)$ is defined as the below ante-derivative of $2\pi K(\lambda \mid \eta)$:

$$\vartheta(\lambda \mid \eta) = 2\pi \int_{\Gamma_\lambda} K(\mu - 0^+ \mid \eta) \cdot d\mu \quad \text{with} \quad \Gamma_\lambda = [0; i\Im(\lambda)] \cup [i\Im(\lambda); \lambda]. \quad (2.10)$$

The -0^+ prescription indicates that the poles of the integrand at $\pm i\eta + i\pi\mathbb{Z}$ should be avoided from the left, *c.f.* Fig. 2.1.2. Throughout the paper, I agree upon $\theta(\lambda) = \vartheta(\lambda \mid \zeta)$.

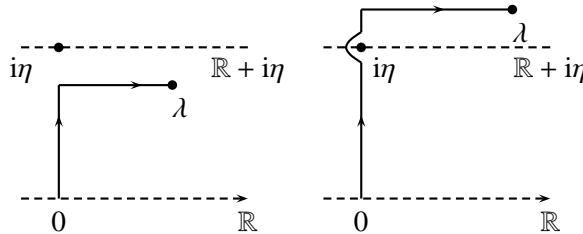


Figure 1: Prescription for the contour Γ_λ plotted for two values of λ having imaginary part below and above η in the case where $0 < \eta < \pi/2$ and $0 < \Im(\lambda) < \pi/2$.

It will be convenient to introduce r, p analogues of the bare phase built up from the kernel $K_{r,p}$:

$$\theta_{r,p}(\lambda) = 2\pi \int_{\Gamma_\lambda} K_{r,p}(\mu - 0^+) \cdot d\mu. \quad (2.11)$$

Just as in (2.10), the contour Γ_λ avoids from the left the poles of the integrand lying on $i\mathbb{R}$. Also, later on, I shall use the notation $\theta(\lambda) = \theta_{1,1}(\lambda)$.

The r, p bare phases arise as a result of summation of bare phases evaluated at so-called string configurations. One has, for any ω where the sum makes sense,

$$\sum_{k=1}^p \theta\left(\omega + i\frac{\zeta}{2}(p+1-2k)\right) = \theta_{p,1}(\omega) + \pi m_p(\zeta) \quad \text{with} \quad m_p(\zeta) = \left(2 - p - \delta_{p,1} + 2\lfloor \zeta \frac{p-1}{2\pi} \rfloor + 2\lfloor \zeta \frac{p+1}{2\pi} \rfloor\right). \quad (2.12)$$

This identity can be established by decomposing the sum as

$$\sum_{k=1}^p \theta\left(\omega + i\frac{\zeta}{2}(p+1-2k)\right) = \lim_{\epsilon \rightarrow 0^+} \{s_1(\epsilon) + s_2(\epsilon)\} \quad (2.13)$$

with

$$s_1(\epsilon) = \sum_{k=1}^p \left\{ \theta\left(\omega + i\epsilon + i\frac{\zeta}{2}(p+1-2k)\right) - \theta\left(i\epsilon + i\frac{\zeta}{2}(p+1-2k)\right) \right\} = 2\pi \int_{\Gamma_\omega} K_{p,1}(\mu - 0^+ + i\epsilon) \quad (2.14)$$

and

$$s_2(\epsilon) = \sum_{k=1}^p \theta\left(i\epsilon + i\frac{\zeta}{2}(p+1-2k)\right). \quad (2.15)$$

The second term can be evaluated as a telescopic sum and by using the definition of the principal value integral. One eventually gets $s_2(\epsilon)|_{\epsilon=0^+} = \pi m_p(\zeta)$ while the $\epsilon \rightarrow 0^+$ limit of the first sum is easily taken.

2.2 The solutions to the Lieb equation

It is well known since the work of Hlthen [18] that solutions to linear integral equation of the type $(\text{id} + \mathbb{K}_{I_Q})[f] = g$ describe the thermodynamic properties of the XXZ chain. Here \mathbb{K}_{I_Q} is the integral operator on $L^2(I_Q)$ acting as

$$\mathbb{K}_{I_Q}[f](\lambda) = \int_{-Q}^Q K(\lambda - \mu) f(\mu) d\mu. \quad (2.16)$$

It is a classical fact, see *e.g.* [12, 55], that the operator $\text{id} + \mathbb{K}_{I_Q}$ is invertible for any $Q \in \mathbb{R}^+$. Its inverse takes the form $\text{id} - \mathbb{R}_{I_Q}$ where the resolvent operator \mathbb{R}_{I_Q} is characterised by its integral kernel $R_{I_Q}(\lambda, \mu)$.

Given some parameter $Q > 0$, the dressed momentum is defined as the solution to the linear integral equation

$$p(\lambda | Q) = \vartheta(\lambda | \zeta/2) - \int_{-Q}^Q \theta(\lambda - \mu) p'(\mu | Q) \cdot \frac{d\mu}{2\pi} \quad (2.17)$$

$$= \vartheta(\lambda | \zeta/2) - \frac{p(Q | Q)}{2\pi} (\theta(\lambda - Q) + \theta(\lambda + Q)) - \int_{-Q}^Q K(\lambda - \mu) p(\mu | Q) \cdot d\mu. \quad (2.18)$$

Given $D \in [0; 1/2]$, there exists a unique [12, 55] $Q_D \in [0; +\infty]$ such that $p(Q_D | Q_D) = \pi D$. The dressed momentum is strictly increasing on \mathbb{R} and strictly decreasing on $\mathbb{R} + i\pi/2$:

$$p'(\lambda | Q) > 0 \quad \text{and} \quad p'(\lambda + i\pi/2 | Q) < 0 \quad \text{for } \lambda \in \mathbb{R}. \quad (2.19)$$

The dressed energy is defined as the solution to the linear integral equation

$$(\text{id} + K_{I_Q})[\varepsilon(* | Q)](\lambda) = \varepsilon(\lambda) \quad \text{with} \quad \varepsilon(\lambda) = h - 2J \sin(\zeta) \theta'(\lambda | \tfrac{1}{2}\zeta). \quad (2.20)$$

The r -string dressed momentum and energy are defined, respectively, by

$$p_r(\omega | Q) = \sum_{\ell=1}^r p\left(\omega + i\frac{\zeta}{2}(r+1-2\ell) | Q\right) \quad (2.21)$$

and

$$\varepsilon_r(\omega | Q) = \sum_{\ell=1}^r \varepsilon\left(\omega + i\frac{\zeta}{2}(r+1-2\ell) | Q\right). \quad (2.22)$$

I refer to Proposition 3.7 for an explanation of the origin of such a denomination. It follows from (2.17) that p'_r admits the integral representation

$$p'_r(\omega | Q) = 2\pi K(\omega | r\frac{\zeta}{2}) - \int_{-Q}^Q K_{r,1}(\omega - s) \cdot p'(s | Q) \cdot ds. \quad (2.23)$$

One can show [38] that the dressed momenta of r -string excitations are non-vanishing

$$|p'_r(\lambda)| > 0 \quad \text{for} \quad \lambda \in \mathbb{R} \cup \{\mathbb{R} + i\pi/2\} \quad (2.24)$$

provided[†] that

- i) $\zeta \in]0; \pi/2[$;
- ii) $\zeta \in]\pi/2; \pi[$ and the additional conditions hold $\begin{cases} \sin(\frac{r-1}{2}\zeta) \sin(\frac{r+1}{2}\zeta) < 0 & \text{if } \lambda \in \mathbb{R} \\ \cos(\frac{r-1}{2}\zeta) \cos(\frac{r+1}{2}\zeta) < 0 & \text{if } \lambda \in \mathbb{R} + i\frac{\pi}{2} \end{cases}$.

The dressed phase is defined as the solution to the linear integral equation

$$\phi(\lambda, \mu | Q) = \frac{1}{2\pi} \theta(\lambda - \mu) - \int_{-Q}^Q K(\lambda - \nu) \phi(\nu, \mu | Q) \cdot d\nu. \quad (2.25)$$

In their turn, its r -sum generalisations are defined as

$$\phi_{r,1}(\lambda, \mu | Q) = \frac{1}{2\pi} \theta_{r,1}(\lambda - \mu) - \int_{-Q}^Q K(\lambda - \nu) \phi_{r,1}(\nu, \mu | Q) \cdot d\nu. \quad (2.26)$$

[†]It is conjectured in [38] that, in fact, the property holds true throughout the regime $\zeta \in]\pi/2; \pi[$

In fact, $\phi_{r,1}$ is related to r sums of the dressed phase similarly to (2.12).

Finally, there is yet another solution of importance to the thermodynamics of the XXZ spin-1/2 chain the so-called dressed charge $Z(\lambda | Q)$ solving

$$(\text{id} + K_{I_Q})[Z(* | Q)](\lambda) = 1. \quad (2.27)$$

The dressed charge is closely related to the dressed phase for the below identities hold [36]:

$$\phi(\lambda, Q | Q) - \phi(\lambda, -Q | Q) + 1 = Z(\lambda | Q) \quad \text{and} \quad 1 + \phi(Q, Q | Q) - \phi(-Q, Q | Q) = \frac{1}{Z(Q | Q)} \quad (2.28)$$

It is clear from the form taken by the linear integral equations that their solutions are holomorphic in some open strip $\mathcal{S}_\delta(\mathbb{R})$ centred around \mathbb{R} .

3 The counting functions

3.1 The ground state roots

Throughout the paper, $\Lambda = \{\lambda_a\}_1^{|\Lambda|}$ will denote the set build up from the Bethe roots describing the model's ground state in the spin $L - 2|\Lambda|$ sector. It was shown in [54] that the ground state Bethe roots are real valued and correspond to a solution to the below logarithmic Bethe equation

$$\frac{\vartheta(\lambda_a | \zeta/2)}{2\pi} - \frac{1}{2\pi L} \sum_{b=1}^{|\Lambda|} \vartheta(\lambda_a - \lambda_b | \zeta) + \frac{|\Lambda| + 1}{2L} = \frac{a}{L} \quad a = 1, \dots, |\Lambda|. \quad (3.1)$$

It was shown in [37] that, for L -large enough and irrespectively of the value of ζ , there exist a unique real valued solution Λ to (3.7). This Λ forms a dense distribution, when $L \rightarrow +\infty$ with $|\Lambda|/L \rightarrow D$, on the interval $[-Q_D; Q_D]$, the Fermi zone of the model.

The value of $q = Q_{D_{gs}}$ of the endpoint of the Fermi zone corresponding to the overall ground state, *viz.* the ground state of $\mathcal{H}_{\Delta,h}$ on \mathfrak{h}_{XXZ} , is fixed uniquely in terms of the magnetic field h : q corresponds to the unique positive solution, *c.f.* [12] for more details, to the equation $\varepsilon(q|q) = 0$ where $\varepsilon(\lambda|Q)$ has been defined in (2.20). In particular, when $h_c > h > 0$, the parameter q runs through $]0; +\infty[$ and the ground state is anti-ferromagnetic. The value of q being fixed by the magnetic field, the thermodynamic limit of the *per* site magnetisation in the ground state is given by $1 - 2D_{gs}$ where D_{gs} is expressed in terms of q as $\pi D_{gs} = p(q | q)$. Note that the allowed range for q implies, *c.f.* [12], that $D_{gs} \in]0; 1/2[$. When specialising the endpoint Q of integration to q in (2.20), the dressed energy is such that [12]

$$\varepsilon(\lambda | q) < 0 \quad \text{on }]-q; q[\quad \text{and} \quad \varepsilon(\lambda | q) > 0 \quad \text{on } \{\overline{\mathbb{R}} \setminus]-q; q[\} \cup \{\overline{\mathbb{R}} + i\pi/2\}. \quad (3.2)$$

The dressed energies r -string excitations with $r \geq 2$ are all strictly positive [38], *i.e.* for any $r \geq 2$, there exists $c_r > 0$ such that

$$\varepsilon_r(\lambda | q) \geq c_r > 0 \quad \text{on} \quad \mathbb{R} \cup \{\mathbb{R} + i\pi/2\} \quad (3.3)$$

this provided[‡] that

$$i) \quad \zeta \in]0; \pi/2[;$$

[‡]Again, it is conjectured that, in fact, (3.3) remains true for $\zeta \in]\pi/2; \pi[$ irrespectively of the auxiliary conditions.

$$ii) \quad \zeta \in]\pi/2; \pi[\text{ and } \begin{cases} \sin(\frac{r-1}{2}\zeta) \sin(\frac{r+1}{2}\zeta) < 0, \text{ and } \sin(r\zeta) < 0 & \text{if } \lambda \in \mathbb{R} \\ \cos(\frac{r-1}{2}\zeta) \cos(\frac{r+1}{2}\zeta) < 0, \text{ and } \sin(r\zeta) > 0 & \text{if } \lambda \in \mathbb{R} + i\frac{\pi}{2} \end{cases}.$$

When L is finite, the ground state $|\Lambda\rangle$ belongs to the sector with $|\Lambda|$ spins down, where the integer $|\Lambda|$ grows with L in such a way that $|\Lambda|/L \rightarrow D_{gs}$. In order to avoid technical complications[†] in the analysis that will follow, I will assume that $|D_{gs} - |\Lambda|/L| = O(L^{-2})$.

Throughout the paper, all functions solving a linear integral equation driven by $\text{id} + K_{I_q}$ will be denoted with the auxiliary argument omitted, *e.g.* $p(\lambda) \equiv p(\lambda | q)$, $\phi(\lambda, \mu) \equiv \phi(\lambda, \mu | q)$. Likewise, the integral operator, resp. its resolvent operator and the associated integral, will be denoted as $\text{id} + K$, resp. $\text{id} - R$ and $R(\lambda, \mu)$.

3.2 The ground state counting function

In order to study the large- L behaviour of various quantities expressed in terms of the Bethe roots, it appears convenient to introduce their counting function following [8, 31]:

$$\widehat{\xi}_\Lambda(\omega) = \frac{1}{2\pi} \vartheta(\omega | \zeta/2) - \frac{1}{2\pi L} \sum_{\lambda \in \Lambda} \vartheta(\omega - \lambda | \zeta) + \frac{|\Lambda| + 1}{2L}. \quad (3.4)$$

The latter satisfies, by construction, $\widehat{\xi}_\Lambda(\lambda_a) = a/L$. The ground state counting function was rigorously characterised in [37].

Theorem 3.1. *There exists $\delta > 0$ such that, for L is large enough,*

- *the function $\widehat{\xi}_\Lambda$ is a biholomorphism from $\mathcal{S}_\delta(\mathbb{R})$ onto its image;*
- *the restriction of $\widehat{\xi}_\Lambda$ to \mathbb{R} is strictly increasing;*
- *for any $\epsilon, C > 0$, there exists $c > 0$ such that $\pm \Im(\widehat{\xi}'_\Lambda(\omega)) > c$ for $|\Re(\omega)| \leq C$ and $\epsilon \leq \pm \Im(\omega) \leq \delta$;*
- *$\widehat{\xi}_\Lambda(\lambda) = \frac{p(\lambda)}{2\pi} (1 + O(L^{-2})) + \frac{|\Lambda| + 1}{2L}$ uniformly on $\mathcal{S}_\delta(\mathbb{R})$.*

These properties ensure that, given two parameters $\tau_L, \tau_R \in]-1/8; 1/8[$, one can define unambiguously two roots \widehat{q}_L and \widehat{q}_R as the unique solution on \mathbb{R} to the equations

$$\widehat{\xi}_\Lambda(\widehat{q}_L) = \frac{\tau_L + 1/2}{L} \quad \text{and} \quad \widehat{\xi}_\Lambda(\widehat{q}_R) = \frac{|\Lambda| + \tau_R + 1/2}{L}. \quad (3.5)$$

The properties enjoyed by $\widehat{\xi}_\Lambda$ ensure that, for some bounded in L constants C_ν , $\nu \in \{L, R\}$,

$$\widehat{q}_R - q = \frac{C_R \tau_R}{L} + O\left(\frac{\tau_R^2}{L^2}\right) \quad \text{and} \quad \widehat{q}_L + q = \frac{C_L \tau_L}{L} + O\left(\frac{\tau_L^2}{L^2}\right). \quad (3.6)$$

The interval $[\widehat{q}_L; \widehat{q}_R]$ can be thought of as a finite volume version of the Fermi zone. The parameters τ_ν , $\nu \in \{L, R\}$, appearing in the definition (3.5) of \widehat{q}_ν will play the role of regularisation parameters as will be explained in Section 3.6 to come.

[†]In principle one could have that $|D_{gs} - |\Lambda|/L| = c/L$ for some $c > 0$. In such a case, there would arise additional terms in the various intermediate expansions obtained in the core of the paper. These are not hard to deal with but would make numerous expressions much bulkier without bringing anything new to the physics. See [52] for a more extensive discussion of this issue.

The ground state counting function allows one to introduce an auxiliary contour \mathcal{C} which passes through \widehat{q}_L and \widehat{q}_R . This contour will play an important role in the analysis. In order to construct \mathcal{C} , one first defines a contour $\widehat{\Gamma}$ according to Fig. 3.2 and then sets $\mathcal{C} = \widehat{\xi}_\Lambda^{-1}(\widehat{\Gamma})$. The parameter $\gamma > 0$ appearing in the definition of $\widehat{\Gamma}$ is taken small enough, in particular such that $\widehat{\Gamma} \subset \widehat{\xi}_\Lambda(\mathcal{S}_\delta(\mathbb{R}))$, but otherwise L -independent.

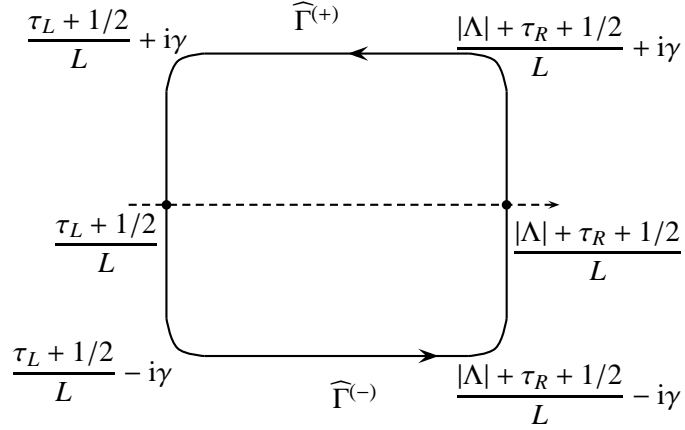


Figure 2: Contour $\widehat{\Gamma} = \widehat{\Gamma}^{(+)} \cup \widehat{\Gamma}^{(-)}$, with $\widehat{\Gamma}^{(\pm)} = \widehat{\Gamma} \cap \mathbb{H}^\pm$. $\gamma > 0$ is a sufficiently small but otherwise fixed constant. The contour \mathcal{C} is defined by $\mathcal{C} = \widehat{\xi}_\Lambda^{-1}(\widehat{\Gamma})$.

3.3 α -twist and b-deformation of the ground-state roots

In the following, I will consider the set of α -twisted, $\alpha \in \mathbb{R}$, solution to the ground state Bethe Ansatz equations, viz. the real valued solution $\Lambda^{(\alpha)} = \{\lambda_a^{(\alpha)}\}_1^{|\Lambda|}$ to $\widehat{\xi}_{\Lambda^{(\alpha)}}(\lambda_a^{(\alpha)}) = a/L$ with $a = 1, \dots, |\Lambda|$, where $\widehat{\xi}_{\Lambda^{(\alpha)}}$ is the associated counting function

$$\widehat{\xi}_{\Lambda^{(\alpha)}}(\omega) = \frac{1}{2\pi} \vartheta(\omega \mid \zeta/2) - \frac{1}{2\pi L} \sum_{\lambda \in \Lambda^{(\alpha)}} \vartheta(\omega - \lambda \mid \zeta) + \frac{|\Lambda| + 1 - 2\alpha_\Lambda}{2L}. \quad (3.7)$$

Here, I assume that $\alpha_\Lambda \in \mathbb{R}$. It can be proven within the techniques of [37] that $\widehat{\xi}_{\Lambda^{(\alpha)}}$ enjoys the conclusion of Theorem 3.1 with the minor difference that

$$\widehat{\xi}_{\Lambda^{(\alpha)}}(\lambda) = \frac{p(\lambda)}{2\pi} (1 + O(L^{-2})) + \frac{|\Lambda| + 1}{2L} - \frac{\alpha_\Lambda}{L} Z(\lambda) \quad \text{uniformly on } \mathcal{S}_\delta(\mathbb{R}). \quad (3.8)$$

In the following, the parameter α_Λ will be assumed to be small enough in L ,

$$\alpha_\Lambda = O(L^{-k_\Lambda}) \quad (3.9)$$

with k_Λ large enough. It will also appear convenient to introduce the b-deformed counting function of the α -twisted ground state roots.

$$\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)} = \widehat{\xi}_{\Lambda^{(\alpha)}} + \frac{b}{L} \quad (3.10)$$

and denote by $\Lambda_b^{(\alpha)}$ the collection of b -deformed ground state roots

$$\Lambda_b^{(\alpha)} = \{\lambda_k^{(\alpha)}(b)\}_{k=1}^{|\Lambda|} \quad \text{with} \quad \lambda_k^{(\alpha)}(b) = \widehat{\xi}_{\Lambda^{(\alpha)}}^{-1}\left(\frac{k-b}{L}\right) \quad k = 1, \dots, |\Lambda|. \quad (3.11)$$

The properties of $\widehat{\xi}_{\Lambda^{(\alpha)}}$ ensure that, for L large enough, the set $\Lambda_b^{(\alpha)}$ is well defined and that all the maps $u \mapsto \lambda_k^{(\alpha)}(u)$ are holomorphic on $\mathcal{D}_{0,1}$.

Proposition 3.2. *Let $\{\mu_a^{(s)}\}_1^{n_{\text{sg}}}$ and $\Upsilon^{(\text{in})}$ be two given sets of real valued parameters. For any $r \leq 1/4$, there exists δr bounded in L and a L -independent constant $C > 0$ such that, for L large enough and for any $b \in \mathbb{C}$ satisfying $|b| = r + \delta r/L$, it holds*

$$d(\Lambda_b^{(\alpha)}, \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}) > \frac{C}{L^2} \quad (3.12)$$

for any α_Λ as in (3.9). If, in addition, one also has $b \notin \mathbb{R}$, then it holds

$$\Lambda_b^{(\alpha)} \cap \{\Upsilon^{(\text{in})} \cup \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}\} = \emptyset. \quad (3.13)$$

In the following, this proposition will be applied to the sets $\Upsilon^{(\text{in})}$ and parameters $\{\mu_a^{(s)}\}_1^{n_{\text{sg}}}$ as defined in Sections 3.4-3.5, equations (3.29)-(3.33), to come.

Proof—

Since $\widehat{\xi}_\Lambda$ preserves the sign of the imaginary part, so does its inverse. Clearly, the same does hold for $\widehat{\xi}_{\Lambda^{(\alpha)}}$. Since the set $\Upsilon^{(\text{in})} \cup \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}$ is real valued, (3.13) clearly holds, this irrespectively of the choice of δr provided that the latter is small enough.

Let G be some relatively compact open neighbourhood of $[-q; q]$ containing all the roots $\Lambda_u^{(0)}$ when u runs through the closed unit disc. It follows from the definition (3.11) of the roots that for any $u \in \mathcal{D}_{0,1}$, these admit the large- L expansion

$$\lambda_a^{(0)}(u) = \lambda_a^{(0)}(0) - \frac{u}{L \widehat{\xi}'_\Lambda(\lambda_a^{(0)}(0))} + \frac{u^2}{2L^2} (\widehat{\xi}_\Lambda^{-1})''\left(\frac{a}{L}\right) + O(L^{-3}). \quad (3.14)$$

Thus, for L is large enough, if $u = |u|e^{i\varphi}$ with $|\sin \varphi| > 1/\sqrt{2}$, one has

$$d(\lambda_a^{(0)}(u), \mathbb{R}) > \frac{|u|}{2\sqrt{2}Lc_\Lambda} \quad \text{with} \quad c_\Lambda = \inf_G |\widehat{\xi}'_\Lambda| > 0. \quad (3.15)$$

Therefore, owing to the real-valuedness of the parameters $\{\mu_a^{(s)}\}_1^{n_{\text{sg}}}$, the lower bound (3.12) automatically holds for such b 's.

Now suppose that $|\cos \varphi| > 1/\sqrt{2}$ and set $\varrho = re^{i\varphi}$. For each k pick a_k such that $|\Re(\lambda_{a_k}^{(0)}(\varrho) - \mu_k^{(s)})|$ is minimal. Then owing to the asymptotic expansion (3.14) and $|\lambda_a^{(0)}(0) - \lambda_{a+1}^{(0)}(0)| \geq \{c_\Lambda L\}^{-1}$, it holds

$$|\Re(\lambda_a^{(0)}(\varrho) - \mu_k^{(s)})| \geq \frac{1}{2Lc_\Lambda} \quad \text{for any } a \neq a_k, \quad (3.16)$$

with c_Λ as given above. For L large enough, given $\ell \leq n_{\text{sg}} + 1$ and

$$b_\ell = \left(r + \frac{\ell}{Lc_\Lambda}\right) \cdot e^{i\varphi} \quad \text{one has} \quad |\Re(\lambda_a^{(0)}(\varrho) - \lambda_a^{(0)}(b_\ell))| \leq \frac{n_{\text{sg}} + 2}{L^2 c_\Lambda^2}. \quad (3.17)$$

The asymptotic expansion (3.14) ensures that if $\ell \neq p$,

$$|\Re(\lambda_a^{(0)}(b_\ell) - \lambda_a^{(0)}(b_p))| > \frac{\cos(\varphi)}{2c_\Lambda^2 L^2} \quad \text{so that} \quad |\Re(\lambda_{a_k}^{(0)}(b_{\ell_k}) - \mu_k^{(s)})| < \frac{1}{4\sqrt{2}c_\Lambda^2 L^2} \quad (3.18)$$

for at most one $\ell_k \in \llbracket 1; n_{\text{sg}} + 1 \rrbracket$. In other words, for any ℓ in the non empty set $\llbracket 1; n_{\text{sg}} + 1 \rrbracket \setminus \{\ell_k\}_1^{n_{\text{sg}}}$, it holds $4\sqrt{2}c_\Lambda^2 L^2 |\Re(\lambda_{a_k}^{(0)}(b_{\ell_k}) - \mu_k^{(s)})| > 1$. Owing to (3.16) and (3.17), this last lower bound holds, with possibly a different constant, not only for a_k , but for any $a = 1, \dots, |\Lambda|$, provided that L is large enough. Finally, since $\alpha_\Lambda = O(L^{-k_\Lambda})$ with k_Λ large enough, one has $\lambda_a^{(0)}(b) - \lambda_a^{(\alpha)}(b) = O(L^{-k_\Lambda})$ so that (3.12) holds for $\Lambda_b^{(\alpha)}$. ■

3.4 The structure of an excited state's Bethe roots

As already stated, within the Bethe Ansatz, one constructs excited states of $H_{\Delta,h}$ as vectors $|\Upsilon\rangle$ parametrised by sets $\Upsilon = \{\mu_a\}_1^{|\Upsilon|}$ build up from solutions to the logarithmic Bethe Ansatz equations

$$\frac{\vartheta(\mu_a | \zeta/2)}{2\pi} - \frac{1}{2\pi L} \sum_{b=1}^{|\Upsilon|} \vartheta(\mu_a - \mu_b | \zeta) + \frac{|\Upsilon| + 1 - 2\alpha_\Upsilon}{2L} = \frac{\ell_a}{L} \quad \text{with} \quad a = 1, \dots, |\Upsilon| \quad (3.19)$$

where $\ell_a \in \mathbb{Z}$ are some integers and where one should set $\alpha_\Upsilon = 0$. The roots μ_a can be real or complex valued [5] but always appear in complex conjugated pairs [51] so that $\Upsilon^* = \Upsilon$. This property remains true as long as α_Υ is real. For a generic state with $|\Upsilon|$ and L arbitrary, the characterisation of the Bethe roots seems extremely difficult -see [6] for a numerical investigation at small $|\Upsilon|$ and L -. However, for excited states close in structure to those of the ground state, namely when $|\Upsilon|/L \rightarrow D_{gs}$ and when Υ mainly consist of real roots forming a dense distribution on $[-q; q]$, one can characterise the Bethe roots completely. First reasonings of the sort go back to Bethe [5]. The arguments raised by Bethe were improved and sophisticated in the works [3, 10, 34, 47, 49]. However, most of the arguments and especially those that were rigorous [3, 49], were concentrated on the sector where $|\Upsilon|$ is fixed and $L \rightarrow +\infty$. The rigorous description of the complex valued solutions in the case described above has only been achieved recently in [38] by the author, this when $0 < D_{gs} < 1/2$. I refer to [38] for a thorough discussion of the history of the subject.

In the following, the set $\Upsilon = \{\mu_a\}_1^{|\Upsilon|}$ will be built out from a solution to the α_Υ -twisted logarithmic Bethe equations (3.19) with real generic α_Υ satisfying to similar bounds as α_Υ viz. $\alpha_\Upsilon = O(L^{-k_\Upsilon})$. Υ will describe an excitation over the ground state at finite magnetic field $h_c > h > 0$. By this I mean that $||\Lambda| - |\Upsilon||$ is fixed and finite in L and that the set $\{\ell_a\}$ differs from $\{1, \dots, |\Lambda|\}$ only by a finite in L number of integers. The conclusion of [38] is that the set Υ can be partitioned as

$$\Upsilon = \{\Upsilon^{(\text{in})} \setminus \Upsilon^{(h)}\} \cup \Upsilon^{(p)} \cup \Upsilon^{(z)}. \quad (3.20)$$

The sets building up this partition are characterised as follows

- $\Upsilon^{(\text{in})} \setminus \Upsilon^{(h)}$ is build out of the real roots contained in Υ that are located inside of the interval $[\widehat{q}_L; \widehat{q}_R]$ and that do not form part of a string of complex roots. In the thermodynamic limit, the elements of $\Upsilon^{(\text{in})}$ form a dense distribution on the Fermi zone $[-q; q]$. $\Upsilon^{(h)}$ is built out of certain roots which form "holes" in this dense distribution. Such roots are called hole roots.
- $\Upsilon^{(p)}$ contains roots belonging to $\{\mathbb{R} \setminus [\widehat{q}_L; \widehat{q}_R]\} \cup \{\mathbb{R} + i\pi/2\}$. Such roots are called particle roots. To avoid complications, the set $\Upsilon^{(p)}$ will be taken to be bounded in L .

- $\Upsilon^{(z)}$ contains the genuinely complex roots. These organise into complexes called strings. Two contiguous elements of a string are separated, up to exponentially small corrections in L , by $i\zeta$. A given string is centred either around \mathbb{R} or around $\mathbb{R} + i\pi/2$. Depending on the parity of its length -odd or even-, a given string may or may not contain an $\mathbb{R} \cup \{\mathbb{R} + i\pi/2\}$ valued root. However, since the complex roots form strings that should be considered as a whole, such $\mathbb{R} \cup \{\mathbb{R} + i\pi/2\}$ valued roots ought to be included in the set $\Upsilon^{(z)}$ rather than be considered as a particle root belonging to $\Upsilon^{(p)}$ or some of the $\Upsilon^{(\text{in})}$ roots. It is convenient to parametrise the set $\Upsilon^{(z)}$ as

$$\Upsilon^{(z)} = \left\{ \left\{ c_a^{(r)} + i\frac{\zeta}{2}(r+1-2k) + \delta_{a,k}^{(r)} \right\}_{k=1}^r \right\}_{a=1}^{n_r^{(z)}} \Bigg\}_{r=2}^{p_{\max}}. \quad (3.21)$$

Within this parametrisation, the centres $c_a^{(r)}$ of the strings belong to $\mathbb{R} \cup \{\mathbb{R} + i\pi/2\}$. The complex roots form strings of length r , with $r = 2, \dots, p_{\max}$. There are $n_r^{(z)}$ different strings of length r , each characterised by the centre $c_a^{(r)}$ with $a = 1, \dots, n_r^{(z)}$. Finally, the parameters $\delta_{a,k}^{(r)}$ represent the so-called string deviations and are exponentially small in L . Without making any explicit emphasis on the decay rate, we shall simply estimate all such corrections as $O(L^{-\infty})$, *viz.* $\delta_{a,k}^{(r)} = O(L^{-\infty})$.

In the following, n_{tot} will refer to the total number of roots differing from the bulk, namely,

$$n_{\text{tot}} = |\Upsilon^{(p)}| + |\Upsilon^{(h)}| + \sum_{r=2}^{p_{\max}} r n_r^{(z)} \quad \text{and} \quad n_{\text{tot}}^{(z)} = \sum_{r=2}^{p_{\max}} r n_r^{(z)}. \quad (3.22)$$

I will always assume in the following that $n_{\text{tot}} = O(\sqrt{L})$.

3.5 The excited state counting function

The various sets arising in the decomposition (3.20) can be characterised more precisely with the help of the counting function subordinate to Υ . As opposed to the case of the ground state roots, some care is needed in defining the latter since the presence of strings of odd length $p \geq 3$ generates a singular behaviour around some points belonging to a shrinking neighbourhood of \mathbb{R} . For such a reason, it is convenient to decompose the counting function into its regular and singular parts

$$\widehat{\xi}_{\Upsilon}(\omega) = \widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega) + \widehat{\xi}_{\Upsilon_{\text{sing}}}(\omega). \quad (3.23)$$

The singular part is built out of the complex roots which collapse to $\mathbb{R} \pm i\zeta$. These roots will be called singular. Since the set Υ is invariant under complex conjugation [51], it is enough to focus on the roots collapsing to $\mathbb{R} + i\zeta$. These are gathered into the set

$$Z^{(s)} = \{z \in \Upsilon : \Im(z) \rightarrow i\zeta\}. \quad (3.24)$$

It is these roots that give rise to the singular part of the counting function:

$$\widehat{\xi}_{\Upsilon_{\text{sing}}}(\omega) = \frac{1}{L} \sum_{\beta + i\zeta \in Z^{(s)}} \int_{\beta^*}^{\beta} \coth(s - \omega) \cdot \frac{ds}{2i\pi} = \frac{1}{2i\pi L} \sum_{\beta + i\zeta \in Z^{(s)}} \ln \left(\frac{\sinh(\beta - \omega)}{\sinh(\beta^* - \omega)} \right). \quad (3.25)$$

The singular part of the counting function has cuts along the segments $[\beta; \beta^*] + i\pi\mathbb{Z}$, $\beta \in Z^{(s)}$ and its derivative

$$\widehat{\xi}_{\Upsilon_{\text{sing}}}^{\prime}(\omega) = \frac{1}{L} \sum_{\beta + i\zeta \in Z^{(s)}} K(\omega - \Re(\beta) | \Im(\beta)) \quad (3.26)$$

may change sign on \mathbb{R} depending on the values of $\Im(\beta)$. Due to the exponential smallness of the string deviation, one has that $\Im(\beta) = O(L^{-\infty})$ and thus $\widehat{\xi}'_{\Upsilon_{\text{sing}}}$ will be exponentially large in a small, $O(L^{-\infty})$ open neighbourhood of $\Re(\beta)$ and exponentially small on $\mathcal{S}_\delta(\mathbb{R})$ provided that one is uniformly away from the set $Z^{(s)} - i\zeta$.

In its turn, the regular part of the counting function is a holomorphic function in some open, L -independent, strip $\mathcal{S}_\delta(\mathbb{R})$, $\delta > 0$, around \mathbb{R} . It is defined as

$$\begin{aligned} \widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega) &= \frac{1}{2\pi} \vartheta(\omega \mid \frac{\zeta}{2}) - \frac{1}{2\pi L} \sum_{\mu \in \Upsilon \setminus \Upsilon^{(z)}} \theta(\omega - \mu) \\ &\quad - \frac{1}{2\pi L} \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \vartheta_{r,a}^{(\text{reg})}(\omega - c_a^{(r)} \mid \{\delta_{a,k}^{(r)}\}) + \frac{|\Upsilon| + 1 - 2\alpha_\Upsilon}{2L}. \end{aligned} \quad (3.27)$$

The function $\vartheta_{r,a}^{(\text{reg})}$ is a regularisation of the total bare phase associated to a given string:

$$\begin{aligned} \vartheta_{r,a}^{(\text{reg})}(\omega \mid \{\delta_{a,k}^{(r)}\}) &= 2\pi \int_{\Gamma_\omega} \left\{ \sum_{k=1}^r K(\mu - 0^+ - i\frac{\zeta}{2}[r+1-2k] - \delta_{a,k}^{(r)}) \right. \\ &\quad \left. - \frac{1}{2i\pi} \cdot \mathbf{1}_Z(c_a^{(r)}) \left[\coth(\mu - 0^+ - \delta_{a, \frac{r-1}{2}}^{(r)}) - \coth(\mu - 0^+ - \delta_{a, \frac{r+3}{2}}^{(r)}) \right] \right\} \cdot d\mu. \end{aligned} \quad (3.28)$$

One has that $\mathbf{1}_Z(c_a^{(r)}) = 1$ if $c_a^{(r)} \in \mathbb{R}$ and $r \geq 3$ is odd and $\mathbf{1}_Z(c_a^{(r)}) = 0$ otherwise. The counter term present in the second line of (3.28) is only there if one deals with a string that contains singular roots. One can check that (3.28) does define a holomorphic function in a neighbourhood of \mathbb{R} .

The set $\Upsilon^{(\text{in})}$ can be defined in terms of the Υ -counting function as

$$\Upsilon^{(\text{in})} = \left\{ \mu \in \text{Int}(\mathcal{C}) : e^{2i\pi L \widehat{\xi}_\Upsilon(\mu)} - 1 = 0 \right\} \setminus \Upsilon^{(z)}. \quad (3.29)$$

In other words, $\Upsilon^{(\text{in})}$ contains all the zeroes of $e^{2i\pi L \widehat{\xi}_\Upsilon} - 1$ lying inside of the contour \mathcal{C} defined in Fig. 2.1.2, with the exception of those zeroes that correspond to an element of a string which squeezes down to $[\widehat{q}_L; \widehat{q}_R]$ in the $L \rightarrow +\infty$ limit. The set $\Upsilon^{(\text{in})}$ contains most of the Bethe roots building up the set Υ . It also contains a certain amount of extra roots, the holes; these are roots of $e^{2i\pi L \widehat{\xi}_\Upsilon} - 1$ located inside of the interval $[\widehat{q}_L; \widehat{q}_R]$ which do not coincide with a root solving the logarithmic Bethe equations (3.19), hence leading to the definition

$$\Upsilon^{(h)} = \Upsilon^{(\text{in})} \setminus \Upsilon = \left\{ \mu \in \text{Int}(\mathcal{C}) : e^{2i\pi L \widehat{\xi}_\Upsilon(\mu)} - 1 = 0 \text{ but } \mu \notin \Upsilon \right\}. \quad (3.30)$$

Finally, the set containing the particle roots can be constructed as

$$\Upsilon^{(p)} = \Upsilon \setminus \left\{ \Upsilon^{(z)} \cup \Upsilon^{(\text{in})} \right\}. \quad (3.31)$$

One can check that the above definitions are indeed consistent with the decomposition (3.20).

It will appear useful, for further handling, to single out a sub-class of singular roots, namely those whose real part is inside of $[\widehat{q}_L; \widehat{q}_R]$:

$$\left\{ \beta_a^{(s)} \right\}_1^{n_{\text{sg}}} = \left\{ z - i\zeta : z \in \Upsilon^{(z)} \text{ satisfying } \Im(z) \rightarrow \zeta \text{ and } \Re(z) \in [\widehat{q}_L; \widehat{q}_R] \right\}. \quad (3.32)$$

Necessarily, a root $\beta_a^{(s)} + i\zeta$ belongs to a string of odd length centred on $[\widehat{q}_L; \widehat{q}_R]$. The central root of such a string will be denoted by $\mu_a^{(s)}$.

The n_{sg} roots $\mu_a^{(s)}$ are then collected into the set

$$\{\mu_a^{(s)}\}_1^{n_{\text{sg}}} = \{z \in \Upsilon^{(z)} : \Im(z) \rightarrow 0 \text{ and } \Re(z) \in [\widehat{q}_L; \widehat{q}_R]\}. \quad (3.33)$$

It is also advantageous to introduce the difference set

$$\daleth = \{\beta_a^{(s)}\}_1^{n_{\text{sg}}} \setminus \{\mu_a^{(s)}\}_1^{n_{\text{sg}}} \quad (3.34)$$

where the parameters $\beta_a^{(s)}$ and $\mu_a^{(s)}$ are defined, respectively, in (3.32) and (3.33). Then according to the convention (1.6) one has

$$\sum_{\alpha \in \daleth} f(\alpha) = \sum_{a=1}^{n_{\text{sg}}} \{f(\beta_a^{(s)}) - f(\mu_a^{(s)})\} \quad \text{and} \quad \prod_{\alpha \in \daleth} f(\alpha) = \prod_{a=1}^{n_{\text{sg}}} \frac{f(\beta_a^{(s)})}{f(\mu_a^{(s)})}. \quad (3.35)$$

To close this section, I state a technical hypothesis that will be used in the analysis of the large- L behaviour. This hypothesis concerns lower bounds in L on the separation of the string centres from the sets $\Upsilon^{(\text{in})} \cup \Upsilon^{(p)}$ as well as on the separation between string centres of equal parity strings and the separation away from zero of the string centres of even length. I will also assume that the set $\Upsilon^{(z)}$ does not contain so-called exact strings, *viz.* strings where some of its constituents are *exactly* spaced by $i\zeta \bmod [i\pi]$. In particular, this means that there are no strings of length r such that $(r-1)\zeta/\pi$ is an integer. Finally, I will assume that the set Υ does not contain repeating elements.

Hypothesis 3.3. *There exists a constants $C > 0$ and $0 \leq \nu < 1/2$ such that*

$$d(\Upsilon^{(\text{in})} \cup \Upsilon^{(p)}, c_a^{(r)} + \delta_{a, \frac{r+1}{2}}^{(r)}) > C \cdot (\Im(\delta_{a, \frac{r-1}{2}}^{(r)}))^{\nu} \quad (3.36)$$

for any choice of the 2-uple (a, r) , $2 \leq r \leq p_{\text{max}}$ and $1 \leq a \leq n_r^{(z)}$, with r being odd.

There exists $C > 0$ and $\kappa > 0$ such that

$$|c_a^{(p)} - c_b^{(r)}| > C \cdot L^{-\kappa} \quad \text{for any } (a, p) \neq (b, r) \quad (3.37)$$

such that p and r have the same parity.

The elements of even length strings are not too close of the points $\pm i\zeta/2$, namely there exists $C > 0$ such that

$$|c_a^{(r)} + \delta_{a, \frac{r}{2}}^{(r)}| > C \left| \delta_{a, \frac{r}{2}}^{(r)} - \delta_{a, \frac{r}{2}+1}^{(r)} \right|^{1/2}, \quad (3.38)$$

for any choice of the 2-uple (a, r) , $2 \leq r \leq p_{\text{max}}$ and $1 \leq a \leq n_r^{(z)}$, with r being odd.

There are no string constituents exactly spaced by $i\zeta$, namely $\delta_{a,k}^{(r)} - \delta_{a,k+1}^{(r)}$, $k = 1, \dots, r-1$ for any a and r . In particular, if $\zeta/\pi \in \mathbb{Q}$ then there are no strings of length r such that $(r-1)\zeta \in \pi\mathbb{Z}$.

The set Υ does not contain repeating elements.

According to the classification that will be established in [38], these hypotheses are satisfied hold for most Υ of interest. In fact, by playing with α_{Υ} small enough and inhomogeneously deforming the model with inhomogeneities $\xi_k = O(L^{-k_{\xi}})$ and k_{ξ} large enough, one can have these assumptions to hold for any state Υ . I will however not discuss these issues further in the present work.

Furthermore, some of the results obtained in the core of the paper are independent of Hypothesis 3.3. It will be made explicit in the statement of a proposition or a theorem whenever this or that assumption will be used in a proof. The lack of exact strings and repeating elements will however be used tacitly in the following.

Using Rouché's theorem, it is an easy corollary of the above hypothesis that, for some $C > 0$ large enough,

$$|\mu_a^{(s)} - \beta_a^{(s)}| \leq C |\Im(\beta_a^{(s)})|^{1-\nu}. \quad (3.39)$$

3.6 The shift function

It will appear convenient in the following to introduce the shift function \widehat{F} associated with the roots $\Lambda^{(\alpha)}$ and Υ

$$\widehat{F}(\omega) = L \left(\widehat{\xi}_{\Lambda^{(\alpha)}}(\omega) - \widehat{\xi}_{\Upsilon}(\omega) \right). \quad (3.40)$$

The shift function is such that

$$e^{2i\pi\widehat{F}(\omega)} = e^{2i\pi\alpha} \prod_{\lambda \in \Lambda^{(\alpha)}} \left\{ \frac{\sinh(\omega - \lambda + i\zeta)}{\sinh(\omega - \lambda - i\zeta)} \right\} \cdot \prod_{\mu \in \Upsilon} \left\{ \frac{\sinh(\omega - \mu - i\zeta)}{\sinh(\omega - \mu + i\zeta)} \right\} = e^{2i\pi\alpha} \frac{V_{\Upsilon; \Lambda^{(\alpha)}}(\omega - i\zeta)}{V_{\Upsilon; \Lambda^{(\alpha)}}(\omega + i\zeta)} \quad (3.41)$$

where

$$V_{\Upsilon; \Lambda^{(\alpha)}}(\omega) = \frac{\prod_{\mu \in \Upsilon} \sinh(\omega - \mu)}{\prod_{\lambda \in \Lambda^{(\alpha)}} \sinh(\omega - \lambda)} \quad \text{and} \quad \alpha = \alpha_{\Upsilon} - \alpha_{\Lambda}. \quad (3.42)$$

It is time to explain the appropriate choice of the parameters $\tau_v \in]-1/8; 1/8[$, $v \in \{L, R\}$. Let κ_v be the integers such that $-1/2 \leq \widehat{F}(\widehat{q}_v) - \kappa_v < 1/2$. The parameters τ_v should be chosen in such a way that

- there exists an L -independent constant $\epsilon_{\Upsilon} > 0$ such that

$$-\frac{1}{2} + \epsilon_{\Upsilon} < \widehat{F}(\widehat{q}_v) - \tau_v - \kappa_v < \frac{1}{2} - \epsilon_{\Upsilon}, \text{ with } v \in \{L, R\}; \quad (3.43)$$

- there exists a constant $C > 0$ such that

$$\min_{\substack{\beta + i\zeta \\ \in Z^{(s)}}} \left\{ d(\beta, \mathcal{C}), d(\beta^*, \mathcal{C}) \right\} > \frac{C}{L}. \quad (3.44)$$

It is clear that there are numerous ways to satisfy the first constraint (3.43). To convince oneself that the second constraint (3.44) can also be fulfilled, one should observe that the large- L behaviour of $\widehat{\xi}_{\Lambda}$ ensures that changing $\tau_v \mapsto \tau'_v$ shifts \widehat{q}_v by a factor $(\tau_v - \tau'_v)/\{L\widehat{\xi}'_{\Lambda}(\widehat{q}_v)\}$ plus some higher order corrections in L^{-1} . Since the singular roots β in (3.44) squeeze with exponential speed on \mathbb{R} , the resulting change in \mathcal{C} is indeed enough so as to fulfil (3.44).

Since $\alpha_{\Lambda} = O(L^{-k_{\Lambda}})$, the property (3.43) allows one to be in a situation where all three functions

$$1 - \exp\{2i\pi L\widehat{\xi}_{\Lambda}\} \quad , \quad 1 - \exp\{2i\pi L\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}\} \quad \text{and} \quad 1 - \exp\{2i\pi L\widehat{\xi}_{\Upsilon}\}$$

are uniformly away from zero at \widehat{q}_v . This non-vanishing is a crucial property for the analysis to come. In particular, it entails that the function $1 - e^{2i\pi L \widehat{\xi}_\Upsilon}$ is non-vanishing on \mathcal{C} , and thus that $\Upsilon^{(\text{in})}$ is well defined. (3.44) allows one to avoid the case when the singular roots will be approaching too close to \widehat{q}_v , what would generate additional problems in the analysis.

It will also appear useful, at some later stage, to introduce the regular part of the shift function,

$$\widehat{F}_{\text{reg}}(\tau) = L \left(\widehat{\xi}_{\Lambda^{(a)}}(\tau) - \widehat{\xi}_{\Upsilon_{\text{reg}}}(\tau) \right). \quad (3.45)$$

3.7 The asymptotic expansion of the Υ -counting function

Below, I establish the large- L expansion of the counting function $\widehat{\xi}_\Upsilon$ on the basis of a certain properties it enjoys. These properties are established in [38] and this demands a separate kind of analysis. Thus, for the purpose of the present paper, they can just be thought of as a set of hypothesis under which the conclusion of the analysis does hold.

- The restriction of $\widehat{\xi}_\Upsilon$ to compact subsets of \mathbb{R} is strictly increasing uniformly in L .
- For any $\epsilon, C > 0$, there exists $c > 0$ such that $\pm \Im(\widehat{\xi}'_\Upsilon(\omega)) > c$ for $|\Re(\omega)| \leq C$ and $\epsilon \leq \pm \Im(\omega) \leq \delta$.
- There exists $C' > 0$ and $\delta > 0$ such that $\|\widehat{F}_{\text{reg}}\|_{S_\delta(\mathbb{R})} < C'$, with \widehat{F}_{reg} as in (3.45).

The strategy to obtain the asymptotic expansion of $\widehat{\xi}_\Upsilon$ consists in writing down a non-linear integral equation (NLIE) satisfied by it [9, 31, 32]. This NLIE can be written precisely because the above properties does hold. Its very form does allow one for an easy calculation of the asymptotic expansion of the counting function. The asymptotic expansion obtained below slightly differs, in structure, from the one obtained in [38] owing to a different choice, more appropriate for the present analysis, of the contour \mathcal{C} in the NLIE satisfied by $\widehat{\xi}_\Upsilon$. More precisely, by changing the definition of such a contour, one changes the definition of "particle" and "hole" root, *viz.* which roots are called "particle" and which ones are called "hole". For two given contours \mathcal{C} and \mathcal{C}' , one can always construct a bijection between the sets $\Upsilon_{\mathcal{C}}^{(p/h)}$ and $\Upsilon_{\mathcal{C}'}^{(p/h)}$, but the latter can quickly become rather complicated.

Prior to stating the result, I introduce a convenient auxiliary function that arises in the intermediate analysis

$$\widehat{u}_\Upsilon(\omega) = \begin{cases} -2i\pi L \cdot [\widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega) + \widehat{\xi}_{\Upsilon_{\text{sing}}}(\omega)] + \widehat{u}_\Upsilon^{(+)}(\omega) & \omega \in \mathbb{H}^+ \\ \widehat{u}_\Upsilon^{(-)}(\omega) & \omega \in \mathbb{H}^- \end{cases} \quad \text{with} \quad \widehat{u}_\Upsilon^{(\epsilon)}(\omega) = \ln \left[1 - e^{2i\pi \epsilon L \widehat{\xi}_\Upsilon(\omega)} \right]. \quad (3.46)$$

Although its Λ counterpart will not be used immediately, I already mention the analogous auxiliary function built from the counting function for the b-deformed ground state roots $\Lambda_b^{(\alpha)}$

$$\widehat{u}_{\Lambda_b^{(\alpha)}}(\omega) = \begin{cases} -2i\pi L \cdot \widehat{\xi}_{\Lambda^{(a)}}^{(b)}(\omega) + \widehat{u}_{\Lambda_b^{(\alpha)}}^{(+)}(\omega) & \omega \in \mathbb{H}^+ \\ \widehat{u}_{\Lambda_b^{(\alpha)}}^{(-)}(\omega) & \omega \in \mathbb{H}^- \end{cases} \quad \text{with} \quad \widehat{u}_{\Lambda_b^{(\alpha)}}^{(\epsilon)}(\omega) = \ln \left[1 - e^{2i\pi \epsilon L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}(\omega)} \right]. \quad (3.47)$$

Also, for further convenience, I agree upon

$$\sigma_R = 1 \quad \text{and} \quad \sigma_L = -1. \quad (3.48)$$

Proposition 3.4. *The regular part of the counting function $\widehat{\xi}_\Upsilon$ can be recast as*

$$\widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega) = \frac{1}{2\pi}p(\omega) - \frac{1}{L}\left(F(\omega) + \alpha_\Lambda Z(\omega)\right) + \Re_N[\widehat{\xi}_\Upsilon](\omega) \quad (3.49)$$

where, setting $\alpha = \alpha_\Upsilon - \alpha_\Lambda$,

$$F(\omega) = \left(\alpha + \frac{|\Lambda| - |\Upsilon|}{2}\right)Z(\omega) + \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \phi_{r;1}(\omega, c_a^{(r)}) + \sum_{\mu \in \Upsilon^{(p)} \setminus \Upsilon^{(h)}} \phi(\omega, \mu) - \sum_{v \in \{L, R\}} \kappa_v \sigma_v \phi(\omega, \sigma_v q). \quad (3.50)$$

The remainder term takes the form

$$\begin{aligned} \Re_N[\widehat{\xi}_\Upsilon](\omega) &= \frac{1}{L} \sum_{a=1}^{n_{\text{sg}}} \left\{ \phi(\omega, \mu_a^{(s)}) - \phi(\omega, \beta_a^{(s)}) \right\} - \sum_{\epsilon = \pm 1} \int_{\mathcal{C}(\epsilon)} R(\omega, s) \cdot \left\{ \frac{\widehat{u}_\Upsilon^{(\epsilon)}(s)}{2i\pi L} - \delta_{\epsilon,+} \widehat{\xi}_{\Upsilon_{\text{sing}}}(s) \right\} \cdot ds \\ &+ \sum_{v \in \{R, L\}} \sigma_v \left\{ \frac{-1}{L} \left[L \widehat{\xi}_{\Upsilon_{\text{sing}}}(\widehat{q}_v) + \widehat{F}(\widehat{q}_v)|_{\alpha_\Lambda=0} - \kappa_v - \tau_v \right] \cdot \left[\phi(\omega, \widehat{q}_v) - \phi(\omega, \sigma_v q) \right] \right. \\ &+ \left. \int_{\sigma_v q}^{\widehat{q}_v} R(\omega, s) \cdot \left[\widehat{\xi}_{\Upsilon_{\text{reg}}}(s) - \widehat{\xi}_{\Upsilon_{\text{reg}}}(\widehat{q}_v) \right] \cdot ds \right\} + \frac{1}{2} \left(D - \frac{|\Lambda|}{L} \right) \cdot \left[\phi(\omega, q) + \phi(\omega, -q) \right] \\ &- \frac{1}{2\pi L} \sum_{p=2}^{p_{\max}} \sum_{a=1}^{n_p^{(z)}} \left(\text{id} - R \right) \left[\vartheta_{p;a}^{(\text{reg})}(* - c_a^{(p)} | \{\delta_{a,k}^{(p)}\}) - \theta_{p,1}(* - c_a^{(p)}) \right] \end{aligned} \quad (3.51)$$

where σ_v has been defined in (3.48), $\text{id} - R$ is the inverse to $\text{id} + K$ and $R(\lambda, \mu)$ stands for the integral kernel of the resolvent. Given $\delta > 0$ small enough and for any k , one has the bounds

$$\left\| \Re_N[\widehat{\xi}_\Upsilon] \right\|_{W_k^\infty(\mathcal{S}_\delta(\mathbb{R}))} = O(L^{-2}) \quad \text{and} \quad \left\| \widehat{F}_{\text{reg}} - F \right\|_{W_k^\infty(\mathcal{S}_\delta(\mathbb{R}))} = O(L^{-1}). \quad (3.52)$$

Below, I only sketch the proof and solely insist on the most important points since these will play a role in the various other estimates carried out in the paper. I refer to [38] for more details.

Proof—

To start with, observe that the function

$$\widehat{u}'_\Upsilon(s) = \frac{2i\pi L \widehat{\xi}'_\Upsilon(s)}{e^{2i\pi L \widehat{\xi}_\Upsilon(s)} - 1} \quad \text{has simple poles at} \quad \begin{cases} \Upsilon^{(\text{in})} \cup \{\mu_a^{(s)}\}_1^{n_{\text{sg}}} & \text{with residue } 1 \\ \{\beta_a^{(s)}\}_1^{n_{\text{sg}}} & \text{with residue } -1 \end{cases} \quad (3.53)$$

and that these are its only poles inside of \mathcal{C} . The regular part of the counting function can thus be recast as

$$\widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega) = \frac{\vartheta(\omega | \zeta/2)}{2\pi} - \oint_{\mathcal{C}} \theta(\omega - s) \widehat{u}'_\Upsilon(s) \frac{ds}{4i\pi^2 L} + \Re^{(1)}(\omega) + \frac{1}{L} \Theta(\omega | \Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) + \frac{|\Upsilon| + 1 - 2\alpha_\Upsilon}{2L} \quad (3.54)$$

where

$$\Theta(\omega | \Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) = \frac{1}{2\pi} \sum_{\mu \in \Upsilon^{(h)} \setminus \Upsilon^{(p)}} \theta(\omega - \mu) - \frac{1}{2\pi} \sum_{p=2}^{p_{\max}} \sum_{a=1}^{n_p^{(z)}} \theta_{p,1}(\omega - c_a^{(p)}) \quad (3.55)$$

and, using the notation (3.35), the remainder takes the form

$$\Re^{(1)}(\omega) = -\frac{1}{2\pi L} \sum_{\mu \in \mathbb{T}} \theta(\omega - \mu) - \frac{1}{2\pi L} \sum_{p=2}^{p_{\max}} \sum_{a=1}^{n_p^{(z)}} \left\{ \vartheta_{p;a}^{(\text{reg})}(\omega - c_a^{(p)} \mid \{\delta_{a,k}^{(p)}\}) - \theta_{p,1}(\omega - c_a^{(p)}) \right\}. \quad (3.56)$$

Upon integrations by part, the contour integral can be recast as

$$\begin{aligned} - \oint_{\mathcal{C}} \theta(\omega - s) \widehat{u}'_{\Upsilon}(s) \frac{ds}{4i\pi^2 L} &= - \int_{\widehat{q}_L}^{\widehat{q}_R} K(\omega - s) \cdot \widehat{\xi}_{\Upsilon_{\text{reg}}}^{(\text{sym})}(s) \cdot ds - \mathfrak{d}[\widehat{u}_{\Upsilon}^{(\text{sym})}](\widehat{q}_R) \cdot \frac{\theta(\omega - \widehat{q}_R)}{2\pi L} \\ &+ \mathfrak{d}[\widehat{u}_{\Upsilon}^{(\text{sym})}](\widehat{q}_L) \cdot \frac{\theta(\omega - \widehat{q}_L)}{2\pi L} - \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} K(\omega - s) \left[\widehat{u}_{\Upsilon}^{(\epsilon)}(s) - 2i\pi L \delta_{\epsilon,+} \widehat{\xi}_{\Upsilon_{\text{sing}}}(s) \right] \cdot \frac{ds}{2i\pi L}. \end{aligned} \quad (3.57)$$

Here, I agree upon

$$\widehat{\xi}_{\Upsilon_{\text{reg}}}^{(\text{sym})} = \widehat{\xi}_{\Upsilon_{\text{reg}}} - (|\Lambda| + 1)/2L, \quad \widehat{u}_{\Upsilon}^{(\text{sym})} = \widehat{u}_{\Upsilon} + i\pi(|\Lambda| + 1)\mathbf{1}_{\mathbb{H}^+} \quad (3.58)$$

and that, given a piecewise continuous function f on \mathcal{C} and continuous on $\mathcal{C}^{(\pm)} = \mathcal{C} \cap \mathbb{H}^{\pm}$, the operator $\mathfrak{d}[f]$ is defined by

$$\mathfrak{d}[f](s) = \frac{-1}{2i\pi} \lim_{\epsilon \rightarrow 0^+} \left\{ f(s + i\epsilon) - f(s - i\epsilon) \right\}. \quad (3.59)$$

By using that, for $\sigma \in [-1/2; 1/2]$, one has

$$\ln[1 + e^{2i\pi\sigma}] - \ln[1 + e^{-2i\pi\sigma}] = 2i\pi\sigma, \quad (3.60)$$

the variations of $\widehat{u}_{\Upsilon}^{(\text{sym})}$ can be recast as

$$\mathfrak{d}[\widehat{u}_{\Upsilon}^{(\text{sym})}](\widehat{q}_v) = L\widehat{\xi}_{\Upsilon}^{(\text{sym})}(\widehat{q}_v) + L\widehat{\xi}_{\Upsilon_{\text{sing}}}(\widehat{q}_v) + \widehat{F}(\widehat{q}_v)|_{\alpha_{\Lambda}=0} - \kappa_v - \tau_v, \quad v \in \{L, R\}. \quad (3.61)$$

After some additional algebra, one gets that

$$- \oint_{\mathcal{C}} \theta(\omega - s) \widehat{u}'_{\Upsilon}(s) \frac{ds}{4i\pi^2 L} = -\mathbb{K}[\widehat{\xi}_{\Upsilon_{\text{reg}}}^{(\text{sym})}](\omega) - \left(\frac{D}{2} - \frac{\kappa_R}{L}\right) \frac{\theta(\omega - q)}{2\pi} - \left(\frac{D}{2} + \frac{\kappa_L}{L}\right) \frac{\theta(\omega + q)}{2\pi} + \Re^{(2)}[\widehat{\xi}_{\Upsilon}](\omega)$$

where the remainder term takes the form

$$\begin{aligned} \Re^{(2)}[\widehat{\xi}_{\Upsilon}](\omega) &= - \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} K(\omega - s) \cdot \left\{ \frac{\widehat{u}_{\Upsilon}^{(\epsilon)}(s)}{2i\pi L} - \delta_{\epsilon,+} \widehat{\xi}_{\Upsilon_{\text{sing}}}(s) \right\} \cdot ds + \left(D - \frac{|\Lambda|}{L}\right) \frac{\theta(\omega - q) + \theta(\omega + q)}{4\pi} \\ &+ \sum_{v \in \{R, L\}} \sigma_v \left\{ \frac{-1}{2\pi L} \left[L\widehat{\xi}_{\Upsilon_{\text{sing}}}(\widehat{q}_v) + \widehat{F}(\widehat{q}_v)|_{\alpha_{\Lambda}=0} - \kappa_v - \tau_v \right] \cdot \left[\theta(\omega - \widehat{q}_v) - \theta(\omega - \sigma_v q) \right] \right. \\ &\quad \left. + \int_{\sigma_v q}^{\widehat{q}_v} K(\omega - s) \cdot \left[\widehat{\xi}_{\Upsilon_{\text{reg}}}(s) - \widehat{\xi}_{\Upsilon_{\text{reg}}}(\widehat{q}_v) \right] \cdot ds \right\}. \end{aligned}$$

Thus, the regular part of the counting function satisfies to the non-linear integral equation

$$\begin{aligned} (\text{id} + \mathbf{K})[\widehat{\xi}_{\text{reg}}^{(\text{sym})}](\omega) &= \frac{\vartheta(\omega \mid \zeta/2)}{2\pi} - \frac{D}{4\pi L}(\theta(\omega - q) + \theta(\omega + q)) + \frac{1}{L}\Theta(\omega \mid \Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) \\ &+ \frac{1}{2\pi L}(\kappa_R\theta(\omega - q) - \kappa_L\theta(\omega + q)) + \frac{|\Upsilon| - |\Lambda| - 2\alpha_\Upsilon}{2L} + \mathfrak{r}^{(1)}(\omega) + \mathfrak{r}^{(2)}[\widehat{\xi}_\Upsilon](\omega). \end{aligned} \quad (3.62)$$

The representation (3.49) follows upon inverting the operator $\text{id} + \mathbf{K}$. The bounds on the remainder $\mathfrak{R}_N[\widehat{\xi}_\Upsilon]$ result from the estimates (3.6) and (B.14) as well as the fact that, for $p \geq 3$,

$$\vartheta_{p;a}^{(s)}(\omega \mid \{\delta_{a,k}^{(p)}\}) = \theta_{p,1}(\omega) + O(\widetilde{\delta}) \quad \text{with} \quad \widetilde{\delta} = \max_{a,k} |\delta_{a,k}^{(p)}| \quad (3.63)$$

where $\theta_{p,1}$ has been defined in (2.11). This allows one to conclude on the negligibility of the remainder. \blacksquare

3.8 Asymptotic expansion of the excitation energy and momentum

The momentum and energy of an Eigenstate $|\Upsilon\rangle$ are given by the expressions [43]

$$\widehat{\mathcal{P}}_\Upsilon = \sum_{\mu \in \Upsilon} \vartheta(\mu \mid \zeta/2) \quad \text{and} \quad \widehat{\mathcal{E}}_\Upsilon = \left(J\Delta - \frac{h}{2}\right)L + \sum_{\mu \in \Upsilon} \mathfrak{e}(\mu) \quad (3.64)$$

where the bare energy \mathfrak{e} is as given in (2.20). The relative excitation energy $\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda}$ and momentum $\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda}$ of an excited state $|\Upsilon\rangle$ in respect to the ground state $|\Lambda\rangle$ are defined as the differences

$$\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda} = \widehat{\mathcal{E}}_\Upsilon - \widehat{\mathcal{E}}_\Lambda \quad \text{and} \quad \widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda} = \widehat{\mathcal{P}}_\Upsilon - \widehat{\mathcal{P}}_\Lambda. \quad (3.65)$$

Below, I establish the large- L expansion of the slightly more general quantities $\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}$ and $\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}$. The latter is a rather direct consequence of the two lemmata below.

Lemma 3.5. *Let f be holomorphic in an neighbourhood of I_q , regular on $\Upsilon_{\text{tot}}^{(z)}$ and assume that there exists $\delta_f > 0$ such that*

$$f' \in L^\infty\left(\mathcal{S}_{\delta_f}\left(\left\{\mathbb{R} + ik\frac{\zeta}{2} + is\frac{\pi}{2}, k \in \llbracket -2(p_{\max} - 1); 2(p_{\max} - 1) \rrbracket, s \in \{0, 1\}\right\}\right) \setminus \mathcal{D}_{\pm i\frac{\zeta}{2}, \frac{1}{2}\delta_f}\right) \quad (3.66)$$

and

$$|f'(z)| \leq \frac{C}{d^\ell(z, \pm i\zeta/2)} \quad \text{on} \quad \mathcal{D}_{\pm i\frac{\zeta}{2}, \delta_f} \setminus \{\pm i\zeta/2\}, \quad (3.67)$$

for some $C > 0$ and $\ell \in \mathbb{N}$. Further, assume that (3.38) holds.

Let

$$\mathcal{S}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}[f] = \sum_{\mu \in \Upsilon} f(\mu) - \sum_{\lambda \in \Lambda_b^{(\alpha)}} f(\lambda). \quad (3.68)$$

Then, it holds

$$\begin{aligned} \mathcal{S}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}[f] &= \left(\alpha + \frac{|\Lambda| - |\Upsilon|}{2} \right) \int_{-q}^q Z(s) f'(s) ds + \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} v_r[f](c_a^{(r)}) + \mathfrak{b} \sum_{v \in \{L, R\}} \sigma_v f(\sigma_v q) \\ &\quad + \sum_{\mu \in \Upsilon^{(p)} \setminus \Upsilon^{(h)}} v_1[f](\mu) - \sum_{v \in \{L, R\}} \sigma_v \mathfrak{K}_v v_1[f](\sigma_v q) + O(L^{-1}) \end{aligned} \quad (3.69)$$

where α is as in (3.42),

$$v_r[f](\omega) = f_r(\omega) - \int_{-q}^q f(s) \partial_s \phi_{r,1}(s, \omega) \cdot ds + \sum_{v \in \{L, R\}} \sigma_v f(\sigma_v q) \phi_{r,1}(\sigma_v q, \omega) \quad (3.70)$$

with

$$f_r(\omega) = \lim_{\epsilon \rightarrow 0^+} \sum_{k=1}^r f\left(\omega + i \frac{\zeta}{2}(r+1-2k) + i\epsilon\right). \quad (3.71)$$

Proof — Given $\Omega \in \{\Upsilon, \Lambda_b^{(\alpha)}\}$, I agree upon

$$\begin{cases} \widehat{\xi}_{\Omega_{\text{reg}}} = \widehat{\xi}_{\Upsilon_{\text{reg}}} \\ \widehat{\xi}_{\Omega_{\text{sing}}} = \widehat{\xi}_{\Upsilon_{\text{sing}}} \end{cases} \quad \text{for } \Omega = \Upsilon \quad \text{and} \quad \begin{cases} \widehat{\xi}_{\Omega_{\text{reg}}} = \widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)} \\ \widehat{\xi}_{\Omega_{\text{sing}}} = 0 \end{cases} \quad \text{for } \Omega = \Lambda_b^{(\alpha)}. \quad (3.72)$$

Then, for such an Ω , by repeating the handlings outlined in the proof of Proposition 3.4 and using that

$$\widehat{u}_{\Omega}^{(+)}(\widehat{q}_v) - \widehat{u}_{\Omega}^{(-)}(\widehat{q}_v) = 2i\pi\tau_v + 2i\pi\delta_{\Omega; \Upsilon}(\mathfrak{K}_v - \widehat{F}(\widehat{q}_v))\big|_{\alpha_{\Lambda}=0} + 2i\pi\mathfrak{f}_v\delta_{\Omega; \Lambda_b^{(\alpha)}} , \quad (3.73)$$

where $\mathfrak{f}_v = L(\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(\widehat{q}_v))$, one gets

$$\begin{aligned} \sum_{\mu \in \Omega} f(\mu) &= L \int_{\widehat{q}_L}^{\widehat{q}_R} \widehat{\xi}'_{\Omega_{\text{reg}}}(s) f(s) \cdot ds - \sum_{v \in \{L, R\}} \sigma_v \left\{ \tau_v + \delta_{\Omega; \Upsilon}(\mathfrak{K}_v - \widehat{F}(\widehat{q}_v))\big|_{\alpha_{\Lambda}=0} + \mathfrak{f}_v \delta_{\Omega; \Lambda_b^{(\alpha)}} \right\} f(\widehat{q}_v) \\ &\quad + \delta_{\Omega; \Upsilon} \sum_{\substack{\mu \in \Upsilon^{(z)} \cup \Upsilon^{(p)} \\ \cup \Upsilon \setminus \Upsilon^{(h)}}} f(\mu) - \sum_{\epsilon = \pm} \int_{\mathcal{C}'(\epsilon)} \left\{ \widehat{u}_{\Omega}^{(\epsilon)}(s) + 2i\pi L \widehat{\xi}'_{\Omega_{\text{sing}}}(s) \right\} f(s) \cdot \frac{ds}{2i\pi}. \end{aligned} \quad (3.74)$$

Here, I agree upon

$$\delta_{\Omega; \Upsilon} = 1 \quad \text{if } \Omega = \Upsilon \quad \text{and} \quad \delta_{\Omega; \Upsilon} = 0 \quad \text{if } \Omega = \Lambda_b^{(\alpha)}. \quad (3.75)$$

The last integral can be estimated, by means of Lemmata B.2 and B.3, to be a $O(L^{-1})$. Likewise, the bound (3.39) ensures that $\sum_{\mu \in \Upsilon} f(\mu) = O(L^{-\infty})$ and (B.7) yields $\widehat{F}(\widehat{q}_v) = \widehat{F}_{\text{reg}}(\widehat{q}_v) + O(n_{\text{sg}} L^{-\infty})$. Thus, one gets that

$$\mathcal{S}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}[f] = - \int_{\widehat{q}_L}^{\widehat{q}_R} f(s) \widehat{F}'_{\text{reg}}(s) ds - \sum_{v \in \{L, R\}} \sigma_v \cdot (\mathfrak{K}_v - \widehat{F}_{\text{reg}}(\widehat{q}_v) - \mathfrak{b}) f(\widehat{q}_v) + \sum_{\mu \in \Upsilon^{(z)} \cup \Upsilon^{(p)} \setminus \Upsilon^{(h)}} f(\mu) + O(L^{-1}). \quad (3.76)$$

It then remains to recall the form (3.49) of the asymptotic expansion of $\widehat{\xi}_{\gamma_{\text{reg}}}$, the bounds (3.6) on the deviations of the endpoints \widehat{q}_v from $\sigma_v q$ and to neglect the exponentially small string deviations what can be achieved by using either the L^∞ property in (3.66) or, in the case of strings of even length, the bounds in (3.67) and hypothesis (3.38). Doing so produces $O(n_{\text{tot}}^{(z)} \cdot L^{-\infty})$ corrections. ■

Lemma 3.6. *Let $t[f]$ be the unique solution to the linear integral equation*

$$(\text{id} + K)[t[f]](\omega) = f(\omega) - \frac{1}{2\pi} \sum_{v \in \{L, R\}} \sigma_v t[f](\sigma_v q) \theta(\omega - \sigma_v q) \quad (3.77)$$

and let $t_r[f](\omega) = \lim_{\epsilon \rightarrow 0^+} \sum_{k=1}^r t[f](\omega + i\frac{\zeta}{2}(r+1-2k) + i\epsilon)$.
Then, for $\omega \in \mathbb{R} \cup \{\mathbb{R} + i\pi/2\}$, it holds

$$v_r[f](\omega) = t_r[f](\omega) + \frac{\widetilde{m}_r(\zeta, \omega)}{2} \cdot \sum_{v \in \{L, R\}} \sigma_v t[f](\sigma_v q), \quad (3.78)$$

with

$$\widetilde{m}_r(\zeta, \omega) = m_r(\zeta) - 2\mathbf{1}_{\mathbb{R} + i\frac{\pi}{2}}(\omega) \left\{ \sum_{\epsilon = \pm} \text{sgn}\left(1 + 2\lfloor \frac{r+\epsilon}{2\pi} \zeta \rfloor - \frac{r+\epsilon}{\pi} \zeta\right) - \delta_{r,1} \right\} \quad (3.79)$$

$m_r(\zeta)$ as defined in (2.12). One also has

$$\int_{-q}^q Z(s) f'(s) \cdot ds = \sum_{v \in \{L, R\}} \sigma_v t[f](\sigma_v q) \quad (3.80)$$

Proof—

By taking derivatives and integrating by parts, one gets that $\partial_s \phi_{r,1}(s, \mu)$ solves the linear integral equation

$$(\text{id} + K)[\partial_s \phi_{r,1}(*, \mu)](\omega) = K_{r,1}(\omega - \mu) + \sum_{v \in \{L, R\}} \sigma_v K(\omega - \sigma_v q) \phi_{r,1}(\sigma_v q, \mu). \quad (3.81)$$

Likewise, owing to (2.12), one readily infers that $t_r[f]$ can be expressed as

$$t_r[f](\omega) = f_r(\omega) - \int_{-q}^q K_{r,1}(\omega - s) t[f](s) ds - \sum_{v \in \{L, R\}} \sigma_v t[f](\sigma_v q) \cdot \left\{ \frac{1}{2\pi} \theta_{r,1}(\omega - \sigma_v q) + \frac{m_r(\zeta)}{2} \right\} \quad (3.82)$$

The first identity in (3.78) then follows upon substituting f and f_r in terms of $t[f]$ and $t_r[f]$ into the definition of $v_r[f]$ and using that $\theta_{r,1}$ is odd on \mathbb{R} while

$$\theta_{r,1}(u) + \theta_{r,1}(-u) - 2\pi\delta_{r,1} = -2\pi \sum_{\epsilon = \pm} \text{sgn}\left(1 + 2\lfloor \frac{r+\epsilon}{2\pi} \zeta \rfloor - \frac{r+\epsilon}{\pi} \zeta\right) \quad \text{when } u \in \mathbb{R} + i\frac{\pi}{2}. \quad (3.83)$$

Finally, the last identity (3.80) is a consequence of $Z = (\text{id} + K)^{-1}[1]$ and the fact that $(\text{id} + K)[(t[f])'] = f'$. ■

One is now in position to establish the large- L expansion of $\widehat{\mathcal{E}}_{\gamma \setminus \Lambda_b^{(\alpha)}}$ and $\widehat{\mathcal{P}}_{\gamma \setminus \Lambda_b^{(\alpha)}}$

Proposition 3.7. *Let $b = \varrho + O(L^{-1})$. One has the large- L expansion*

$$\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda_b^{(\alpha)}} = \sum_{\mu \in \Upsilon^{(p)}} \varepsilon(\mu) - \sum_{\mu \in \Upsilon^{(h)}} \varepsilon(\mu) + \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \varepsilon_r(c_a^{(r)}) + O(L^{-1}) \quad (3.84)$$

and, with α as in (3.42),

$$\begin{aligned} \widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda_b^{(\alpha)}} &= \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} p_r(c_a^{(r)}) + \sum_{\mu \in \Upsilon^{(p)} \setminus \Upsilon^{(h)}} p(\mu) + 2\varrho\theta(q \mid \frac{\zeta}{2}) \\ &+ \left(2\alpha + |\Lambda| - |\Upsilon| + \sum_{p=2}^{p_{\max}} \sum_{a=1}^{n_p^{(z)}} \widetilde{m}_p(\zeta, c_a^{(p)}) + \sum_{\mu \in \Upsilon^{(p)}} \widetilde{m}_1(\zeta, \mu) - \kappa_L - \kappa_R\right) p(q) + O(L^{-1}) \end{aligned} \quad (3.85)$$

where $\widetilde{m}_p(\zeta, \omega)$ is as defined by (3.79).

Proof —

First observe that $\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda_b^{(\alpha)}} = \mathcal{S}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}[\mathfrak{e}]$ and $\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda_b^{(\alpha)}} = \mathcal{S}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}[\vartheta(* \mid \zeta/2)]$ where $\mathcal{S}_{\Upsilon \setminus \Lambda}$ is as defined in (3.69). Since both $\vartheta(* \mid \zeta/2)$ and \mathfrak{e} satisfy to the property (3.66), Lemmata 3.5 and 3.6 guarantee that one has

$$\begin{aligned} \mathcal{S}_{\Upsilon \setminus \Lambda_b^{(\alpha)}}[f] &= \sum_{v \in \{L, R\}} \sigma_v \left(\alpha + \frac{|\Lambda| - |\Upsilon|}{2} + \frac{1}{2} \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \widetilde{m}_r(\zeta, c_a^{(r)}) + \frac{1}{2} \sum_{\mu \in \Upsilon^{(p)}} \widetilde{m}_1(\zeta, \mu) - \kappa_v \right) \cdot \mathfrak{t}[f](\sigma_v q) \\ &+ b \sum_{v \in \{L, R\}} \sigma_v f(\sigma_v q) + \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \mathfrak{t}_r[f](c_a^{(r)}) + \sum_{\mu \in \Upsilon^{(p)} \setminus \Upsilon^{(h)}} \mathfrak{t}[f](\mu) + O(L^{-1}). \end{aligned} \quad (3.86)$$

It then only remains to observe that

$$\mathfrak{t}[\vartheta(* \mid \zeta/2)](\mu) = p(\mu) \quad \text{and} \quad \mathfrak{t}[\mathfrak{e}](\mu) = \varepsilon(\mu) \quad (3.87)$$

so as to conclude. \blacksquare

3.9 Parametrisation in the thermodynamic limit

Proposition 3.7 shows that the particle, hole and string roots correspond to different kinds of excitations in the model. Owing to (3.3) one concludes that the strings correspond to various kinds of massive excitations. Since ε changes sign at $\pm q$, *c.f.* (3.2), and that $|\varepsilon(\lambda)| > c > 0$ uniformly away from $\pm q$ on $\mathbb{R} \cup \{\mathbb{R} + i\pi/2\}$, one infers that the particle and hole excitations can give rise to massive and massless excitations depending on the proximity of their parameters to the endpoints of the Fermi zone. If a particle or hole root collapses, when $L \rightarrow +\infty$, on the Fermi boundary then it will produce a vanishing, in the thermodynamic limit, contribution to the excitation energy of that state. However, if it stays at finite distance from the Fermi zone, then owing to (3.2), it will generate a finite positive contribution and hence correspond to a massive excitation.

Massive and massless excitation contribute rather differently to the large- L behaviour of a form factor. Thus, it is convenient to distinguish between such excitations and decompose the particle and hole sets $\Upsilon^{(p)}$ and $\Upsilon^{(h)}$ as

$$\Upsilon^{(p)} = \Upsilon_{\text{off}}^{(p)} \cup \Upsilon_R^{(p)} \cup \Upsilon_L^{(p)} \quad \text{and} \quad \Upsilon^{(h)} = \Upsilon_{\text{off}}^{(h)} \cup \Upsilon_R^{(h)} \cup \Upsilon_L^{(h)}. \quad (3.88)$$

The roots contained in $\Upsilon_{R/L}^{(p)}$ and $\Upsilon_{R/L}^{(h)}$ generate the massless excitations in the model. These roots correspond to the solutions to $\widehat{\xi}_\Upsilon(\mu) = m/L$ where

$$m \in \begin{cases} \lfloor L\widehat{\xi}_\Upsilon(\widehat{q}_v) \rfloor + \sigma_v\{1, 2, \dots\} & \text{if } \mu \in \Upsilon_v^{(p)} \\ \lfloor L\widehat{\xi}_\Upsilon(\widehat{q}_v) \rfloor - \sigma_v\{0, 1, \dots\} & \text{if } \mu \in \Upsilon_v^{(h)} \end{cases}, \quad v \in \{L, R\}. \quad (3.89)$$

In other words, the roots giving rise to the massless excitations can be parametrised by integers $p_a^\nu, h_a^\nu \in \mathbb{N}$, $v \in \{L, R\}$, so that they solve

$$\widehat{\xi}_\Upsilon(\mu) = \frac{1}{L}(|\Lambda| - \kappa_R + p_a^R + 1) \quad \text{for } \mu \in \Upsilon_R^{(p)}, \quad \widehat{\xi}_\Upsilon(\mu) = \frac{1}{L}(|\Lambda| - \kappa_R - h_a^R) \quad \text{for } \mu \in \Upsilon_R^{(h)} \quad (3.90)$$

and

$$\widehat{\xi}_\Upsilon(\mu) = \frac{1}{L}(-\kappa_L - p_a^L) \quad \text{for } \mu \in \Upsilon_L^{(p)}, \quad \widehat{\xi}_\Upsilon(\mu) = \frac{1}{L}(-\kappa_L + h_a^L + 1) \quad \text{for } \mu \in \Upsilon_L^{(h)}. \quad (3.91)$$

Here, the integers κ_L, κ_R are precisely those arising in (3.43). This representation implies that, for any $\mu \in \Upsilon_v^{(p/h)}$,

$$|\mu - \sigma_v q| = O\left(\frac{|k_a^\nu|}{L}\right) \quad (3.92)$$

where k_a^ν is the integer arising in the definition (3.90) or (3.91) of the root μ . Note that κ_v does not arise in the remainder (3.92) owing to the condition (3.43) and hypothesis (3.36) which ensures that for any $\mu \in \Upsilon_{R/L}^{(p/h)}$ one has $\widehat{F} = \widehat{F}_{\text{reg}} + O(n_{\text{sg}} L^{-\infty})$.

The off-boundary roots $\Upsilon_{\text{off}}^{(p/h)}$ generate massive particle-hole excitations. They solve the equation $\widehat{\xi}_\Upsilon(\mu) = m/L$ where m is such that $\lim(m/L) \notin \{0, D\}$. Thus, such roots are located uniformly away from $[-q; q]$.

For further purposes, it will appear convenient to introduce the shorthand notations for the number and for the discrepancies between the number of particles and holes in the left, right and off-boundary collections of roots:

$$n_v^{(p/h)} = |\Upsilon_v^{(p/h)}| \quad \text{and set} \quad \ell_v = |\Upsilon_v^{(p)}| - |\Upsilon_v^{(h)}| \quad \text{for } v \in \{L, R, \text{off}\}. \quad (3.93)$$

It will also appear useful, in the following, to introduce a parametrisation allowing one to interpret the off-critical particle roots as one-strings having no-string deviations:

$$\Upsilon_{\text{off}}^{(p)} = \left\{ c_a^{(1)} \right\}_{a=1}^{n_z^{(1)}} \quad \text{with} \quad n_z^{(1)} = |\Upsilon^{(p)}| \quad \text{and} \quad \delta_{a,1}^{(1)} = 0 \quad \text{for any } a. \quad (3.94)$$

This allows to collect all the string centres associated with all possible lengths $r = 1, \dots, p_{\text{max}}$ into a single set

$$\mathfrak{C} = \left\{ \left\{ c_a^{(p)} \right\}_{a=1}^{n_p^{(z)}} \right\}_{p=1}^{p_{\text{max}}}. \quad (3.95)$$

I stress that \mathfrak{C} does not contain the particle roots which collapse on either of the two edges of the Fermi zone.

When specialising to excited states having the decomposition (3.88), the regular part of the counting function admits the asymptotic expansion

$$\widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega) = \frac{1}{2\pi} p(\omega) - \frac{1}{L} F_\infty(\omega) + O\left(\frac{\mathfrak{U}}{L^2}\right) \quad (3.96)$$

in which

$$F_\infty(\omega) = \left(\alpha + \frac{|\Lambda| - |\Upsilon|}{2}\right) Z(\omega) + \sum_{p=2}^{p_{\max}} \sum_{a=1}^{n_p^{(z)}} \phi_{p;1}(\omega, c_a^{(p)}) + \sum_{\mu \in \Upsilon_{\text{off}}^{(p)} \setminus \Upsilon_{\text{off}}^{(h)}} \phi(\omega, \mu) + \sum_{v \in \{L, R\}} \ell_v^\omega \phi(\omega, \sigma_v q) \quad (3.97)$$

represents the thermodynamic limit of the shift function in presence of such excitations while the remainder is uniform and holomorphic in some fixed strip around \mathbb{R} . The control term in the remainder is given by

$$\mathfrak{U} = \sum_{v \in \{L, R\}} \left\{ \sum_{a=1}^{n_v^{(p)}} (p_a^v + 1/2) + \sum_{a=1}^{n_v^{(h)}} (h_a^v + 1/2) \right\}. \quad (3.98)$$

Here and in the following, it will be tacitly assumed that $\mathfrak{U}/L \leq 1$.

The thermodynamic shift function F_∞ is written, among other things, in terms of the Fermi boundary Umklapp integers

$$\ell_v^\omega = \ell_v - \sigma_v \kappa_v \quad (3.99)$$

that are expressed in terms of the integers ℓ_v and κ_v introduced, respectively, in (3.93), (3.43).

Note that (3.97) entails the bound

$$\|F^{(q)}\|_{W_k(S_\delta(\mathbb{R}))} \leq C n_{\text{tot}}^{(\text{msv})} \quad \text{so that} \quad |\kappa_v| \leq C n_{\text{tot}}^{(\text{msv})}, \quad (3.100)$$

where

$$n_{\text{tot}}^{(\text{msv})} = |\Upsilon_{\text{off}}^{(p)}| + |\Upsilon_{\text{off}}^{(h)}| + n_{\text{tot}}^{(z)} + |\ell_L^\omega| + |\ell_R^\omega| + 1. \quad (3.101)$$

In the following, the number $n_{\text{tot}}^{(\text{msv})}$ will be taken to be bounded in L .

The excitation energy and momentum of such excited states takes the form

$$\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda_b^{(a)}} = \mathcal{P}_{\text{ex}}^{(q)} + \frac{2\pi}{L} \sum_{v \in \{L, R\}} \sigma_v \left\{ \sum_{a=1}^{n_v^{(p)}} p_a^v + \sum_{a=1}^{n_v^{(h)}} (h_a^v + 1) \right\} + \mathcal{O}\left(\frac{1}{L}, \frac{\mathfrak{U}^2}{L^2}\right) \quad (3.102)$$

$$\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda_b^{(a)}} = \mathcal{E}_{\text{ex}} + \frac{2\pi v_F}{L} \sum_{v \in \{L, R\}} \left\{ \sum_{a=1}^{n_v^{(p)}} p_a^v + \sum_{a=1}^{n_v^{(h)}} (h_a^v + 1) \right\} + \mathcal{O}\left(\frac{1}{L}, \frac{\mathfrak{U}^2}{L^2}\right) \quad (3.103)$$

where $v_F = \epsilon'(q)/p'(q)$ is the Fermi velocity,

$$\begin{aligned} \mathcal{P}_{\text{ex}}^{(q)} &= \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} p_r(c_a^{(r)}) + \sum_{\mu \in \Upsilon_{\text{off}}^{(p)} \setminus \Upsilon_{\text{off}}^{(h)}} p(\mu) + 2\varrho \theta(q \mid \tfrac{\zeta}{2}) \\ &+ \left(\sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \widetilde{m}_r(\zeta, c_a^{(r)}) + \sum_{\mu \in \Upsilon_{\text{off}}^{(p)}} \widetilde{m}_1(\zeta, \mu) + |\Lambda| - |\Upsilon| + 2\alpha + \ell_R^\omega - \ell_L^\omega \right) p(q) \end{aligned} \quad (3.104)$$

and

$$\mathcal{E}_{\text{ex}} = \sum_{r=2}^{p_{\max}} \sum_{a=1}^{n_r^{(z)}} \varepsilon_r(c_a^{(r)}) + \sum_{\mu \in \Upsilon_{\text{off}}^{(p)} \setminus \Upsilon_{\text{off}}^{(h)}} \varepsilon(\mu) . \quad (3.105)$$

Above, the $O(1/L, \mathfrak{U}^2/L^2)$ symbol means that the corrections are either of order $O(1/L)$ and do not depend on the integers p_a^ν and h_a^ν or are of order $O(1/L^2)$ but with at most a quadratic dependence on the integers. *Per se* the statement is not a direct consequence of Proposition 3.7 in that one should push the asymptotic expansion of $\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda_b^{(a)}}$ and $\widehat{\mathcal{E}}_{\Upsilon \setminus \Lambda_b^{(a)}}$ obtained there one order further, namely up to $O(L^{-2})$. This is technically involved but does not present any major conceptual difficulty. We thus state the result without sketching its proof.

3.10 The ϱ -regularised shift function

For technical purposes, it appears convenient to introduce the \mathfrak{b} -regularised shift function:

$$\widehat{F}^{(\mathfrak{b})} = L(\widehat{\xi}_{\Lambda^{(a)}}^{(\mathfrak{b})} - \widehat{\xi}_\Upsilon) = \widehat{F} + \mathfrak{b} \quad \text{and} \quad \widehat{F}_{\text{reg}}^{(\mathfrak{b})} = \widehat{F}_{\text{reg}} + \mathfrak{b} . \quad (3.106)$$

Quite analogously, one defines ϱ -regularisations $F^{(\varrho)}$ and $F_\infty^{(\varrho)}$ of the functions F (3.50) and F_∞ (3.97) introduced earlier on

$$F^{(\varrho)} = F + \varrho \quad \text{and} \quad F_\infty^{(\varrho)} = F_\infty + \varrho . \quad (3.107)$$

The main advantage of the ϱ -deformation is that the zeroes of the functions

$$1 - e^{2i\pi F_\infty^{(\varrho)}}, \quad 1 - e^{2i\pi F^{(\varrho)}} \quad \text{and} \quad 1 - e^{2i\pi \widehat{F}^{(\mathfrak{b})}} . \quad (3.108)$$

have nicer properties. Characterising these will be the aim of the lemma below.

Lemma 3.8. *Assume that hypothesis (3.37) holds. Let $F_\infty|_{\alpha=0}$ be the thermodynamic limit of the shift function associated with the sets of parameters $(\mathfrak{C}, \Upsilon_{\text{off}}^{(p)}, \ell_v^{(\mathfrak{x})})$. There exist*

- *an open neighbourhood \mathcal{V}_F of $[-q; q]$ in \mathbb{C}*
- *parameters $\widetilde{\varepsilon} > 0$ and $1/4 > r_1 > r_2 > 0$*

such that,

- *for any shift function satisfying $\|F^{(\varrho)} - F_\infty|_{\alpha=0}\|_{L^\infty(\mathcal{V}_F)} < \widetilde{\varepsilon}$*
- *for any $\varrho, \mathfrak{b} \in \mathbb{C}$ such that $|\varrho| = r$, for some $r \in [r_2; r_1]$ and $\varrho - \mathfrak{b} = O(L^{-1})$*

one has

$$\text{i) } d(\partial \mathcal{V}_F, Z^{(s)} - i\zeta) > C' \text{ for some } C' > 0, \mathcal{V}_F \cap \{Z^{(s)} - i\zeta\} = \{\beta_a^{(s)}\}_1^{n'_{\text{sg}}} \text{ with} \\ \beta_a^{(s)} \in \text{Int}(\mathcal{C}) \text{ for } a = 1, \dots, n_{\text{sg}} \quad \text{and} \quad \beta_a^{(s)} \in \text{Ext}(\mathcal{C}) \text{ for } a = n_{\text{sg}} + 1, \dots, n'_{\text{sg}} . \quad (3.109)$$

ii) The functions $t \mapsto e^{2i\pi F_\infty^{(\varrho)}(t)}|_{\alpha=0}$ and $t \mapsto e^{2i\pi F^{(\varrho)}(t)}$ are both holomorphic and non-zero on \mathcal{V}_F . The function $t \mapsto e^{2i\pi \widehat{F}^{(\mathfrak{b})}(t)}$ is meromorphic on \mathcal{V}_F . Its only zeroes and poles inside on \mathcal{V}_F are the simple poles at $\beta_a^{(s)}$ and the simple zeroes at $(\beta_a^{(s)})^$ with $a = 1, \dots, n'_{\text{sg}}$.*

iii) The function $t \mapsto 1 - e^{2i\pi\widehat{F}^{(b)}(t)}$ admits $n'_{\text{sg}} + \ell_F$ zeroes in \mathcal{V}_F with $\ell_F = O(n_{\text{tot}}^{(\text{msv})})$ and $n_{\text{tot}}^{(\text{msv})}$ as given by (3.101). The zeroes partition as

$$Z = \{\mathfrak{z}_a^{(s)}\}_1^{n'_{\text{sg}}} \quad \text{and} \quad W = \{w_a\}_{a=1}^{\ell_F}. \quad (3.110)$$

The zeroes w_a are simple and well separated: there exists $\epsilon > 0$ such that

$$|w_a - w_b| > 2\epsilon \quad \text{if } a \neq b \quad \text{and} \quad d(W, \{Z^{(s)} - i\zeta\} \cup \Upsilon_{\text{off}}^{(h)} \cup \{\pm q\}) > 2\epsilon. \quad (3.111)$$

The zeroes $\mathfrak{z}_a^{(s)}$ are all simple and there exists $C > 0$ such that

$$C^{-1}|\Im(\beta_a^{(s)})| \leq |\mathfrak{z}_a^{(s)} - \beta_a^{(s)}| \leq C|\Im(\beta_a^{(s)})| \quad \text{and} \quad C^{-1} \leq \left| \frac{\mathfrak{z}_a^{(s)} - (\beta_a^{(s)})^*}{\mathfrak{z}_a^{(s)} - \beta_a^{(s)}} \right| \leq C. \quad (3.112)$$

iv) There exists $\delta > 0$ and $c > 0$ such that

$$\min_{t \in \mathcal{C}} \left\{ |1 - e^{2i\pi F^{(\varrho)}(t)}|, |1 - e^{2i\pi\widehat{F}^{(b)}(t)}| \right\} \geq c \quad \text{and} \quad \|\widehat{F}^{(b)} - F^{(\varrho)}\|_{W_k^\infty(\mathcal{C})} = O(L^{-1}). \quad (3.113)$$

The contour \mathcal{C} depends on δ according to (A.18). Furthermore, given ϵ as in point ii), for each $w \in W$, there exists an L -independent z_w such that w is the only zero contained in $\mathcal{D}_{z_w, \epsilon}$ and

$$\min_{t \in \partial\mathcal{V}_F \cup_{w \in W} \partial\mathcal{D}_{z_w, \epsilon}} \left\{ |1 - e^{2i\pi F^{(\varrho)}(t)}|, |1 - e^{2i\pi\widehat{F}^{(b)}(t)}| \right\} \geq c. \quad (3.114)$$

Note that the roots $\beta_a^{(s)}$, for $a = 1, \dots, n_{\text{sg}}$ are precisely the roots appearing in (3.32). The roots $\beta_a^{(s)}$ with $a = n_{\text{sg}} + 1, \dots, n'_{\text{sg}}$ are singular roots whose real part lies outside of $[\widehat{q}_L; \widehat{q}_R]$ and which are contained in \mathcal{V}_F .

In fact, the lemma can be applied to shift-functions $\widehat{F}^{(b)}$ and $F^{(\varrho)}$ that are not necessarily built from configurations of roots $(\Upsilon^{(z)}, \Upsilon^{(p)}, \Upsilon^{(h)})$ of an excited state which approach the configuration $(\mathfrak{C}, \Upsilon_{\text{off}}^{(p)}, \ell_v^{(z)})$. It is enough that the associated shift function is not too far away from a given thermodynamic limit $F_\infty^{(\varrho)}|_{\alpha=0}$.

Proof —

The properties of the dressed charge and phases ensure that $F_\infty^{(\varrho)}|_{\alpha=0}$ given by (3.97), and hence $1 - e^{2i\pi F_\infty^{(0)}|_{\alpha=0}}$, is holomorphic on a sufficiently small neighbourhood of $[-q; q]$. Therefore, $1 - e^{2i\pi F_\infty^{(0)}|_{\alpha=0}}$ will only have isolated zeroes there. One can thus always pick an open neighbourhood \mathcal{V}_F that is relatively compact, such that the function does not vanish on its boundary and such that the only zeroes contained in \mathcal{V}_F are real and located on $[-q; q]$. It is also clear that one can choose \mathcal{V}_F such that $d(\partial\mathcal{V}_F, Z^{(s)} - i\zeta) > C'$ for some constant C' .

The statements of point ii) are evident.

To establish iii), let u_a with $a = 1, \dots, \ell_F^{(\infty)}$ be the zeroes of $1 - e^{2i\pi F_\infty^{(0)}|_{\alpha=0}}$ in \mathcal{V}_F and let m_a denote the multiplicity of u_a . In virtue of the local behaviour of holomorphic functions, there exist $\varsigma, \varsigma' > 0$, an integer n_k and a biholomorphism g_k

$$g_k : \begin{cases} \mathcal{D}_{u_k, \varsigma} & \rightarrow & g_k(\mathcal{D}_{u_k, \varsigma}) \supset \mathcal{D}_{0, \varsigma'} \\ u_k & \mapsto & 0 \end{cases} \quad \text{such that } F_\infty^{(0)}|_{\alpha=0} = n_k + \frac{1}{2i\pi}(g_k)^{m_k} \quad (3.115)$$

on $\mathcal{D}_{u_k, \varsigma}$. It thus follows that, for any ϱ small enough, $1 - e^{2i\pi F_\infty^{(\varrho)}|_{\alpha=0}}$ admits m_k simple zeroes $u_{k,r}$, $r = 1, \dots, m_k$, on $\mathcal{D}_{u_k, \varsigma}$ which satisfy $|u_{k,r} - u_{k,r'}| > C|\varrho|^{1/m_k}$ for some constant C if $r \neq r'$.

For an arbitrary choice of $|\varrho|$, the roots $u_{k,r}$ may come arbitrarily close to the set $\{Z^{(s)} - i\zeta\} \cup \Upsilon_{\text{off}}^{(h)} \cup \{\pm q\}$ provided that the latter has a non-zero intersection with $\mathcal{D}_{w_k, S}$. In such a case, denote $\{v_{k,r}\}_{r=1}^{d_a}$ the points of intersection:

$$\left(\{Z^{(s)} - i\zeta\} \cup \Upsilon_{\text{off}}^{(h)} \cup \{\pm q\}\right) \cap \mathcal{D}_{u_k, S} = \{v_{k,r}\}_{r=1}^{d_a} \quad (3.116)$$

The roots $u_{k,r}$ will be uniformly away from the set $\{Z^{(s)} - i\zeta\} \cup \Upsilon_{\text{off}}^{(h)} \cup \{\pm q\}$ as long as $|\varrho|$ is at finite distance from the set $\{|g_k(v_{k,r})|^{m_k}, 1 \leq k \leq \ell_F^{(\infty)} \text{ and } 1 \leq r \leq m_k\}$. This can be always done since one deals with a finite collection of points.

As a consequence, there exists $\epsilon > 0$ and $|\varrho| > 0$ small enough such that all the zeroes of $1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0}$ in \mathcal{V}_F are simple, at least distant by 4ϵ from each other, and such that any zero w satisfies

$$d(w, \{Z^{(s)} - i\zeta\} \cup \Upsilon_{\text{off}}^{(h)} \cup \{\pm q\}) > 3\epsilon. \quad (3.117)$$

One can even pick r_1, r_2 such that this property does hold uniformly in $r_1 > |\varrho| > r_2$.

Let $Z_F^{(\infty)} = \{u_{a,k}\}$ be the collection of these simple zeroes and set $\ell_F = |Z_F^{(\infty)}|$. Lemma A.1 applied to the function $1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0}$ on the compact set $\overline{\mathcal{V}_F}$ and some larger, fixed simply connected domain U containing $\overline{\mathcal{V}_F}$ such that this function is holomorphic on \overline{U} and has no zeroes on ∂U ensures that

$$\ell_F \leq C \frac{\ln \left\| 1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0} \right\|_{L^\infty(\mathcal{V}_F)}}{\ln \left\| 1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0} \right\|_{L^\infty(U)}} \leq C' n_{\text{tot}}^{(\text{msv})}. \quad (3.118)$$

The last bound follows from (3.100).

The maximum principle applied to $\left\{1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0}\right\}^{-1}$ on $\mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \mathcal{D}_{z,\epsilon}$ ensures that there exist $c > 0$ such that

$$\left| 1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0} \right| > 2c \quad \text{on} \quad \partial \mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \partial \mathcal{D}_{z,\epsilon}. \quad (3.119)$$

There exists $\tilde{\epsilon} > 0$ small enough, such that for any $F^{(\varrho)}$ satisfying $\|F^{(\varrho)} - F_\infty^{(\varrho)}|_{\alpha=0}\|_{L^\infty(\mathcal{V}_F)} < \tilde{\epsilon}$ it holds

$$\left| e^{2i\pi F^{(\varrho)}} - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0} \right| < \frac{c}{2} \quad \text{on} \quad \partial \mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \partial \mathcal{D}_{z,\epsilon}. \quad (3.120)$$

By applying Rouché's theorem first on \mathcal{V}_F and then on the discs $\mathcal{D}_{z,\epsilon}$ to the function $1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0}$ one obtains that $1 - e^{2i\pi F_\infty^{(\varrho)}}|_{\alpha=0}$ will have $\ell_F = |Z_F^{(\infty)}|$ simple zeroes on \mathcal{V}_F and that any such zero will belong to a unique disk $\mathcal{D}_{z,\epsilon}$ for some $z \in Z_F^{(\infty)}$. Furthermore, by applying the maximum principle to the inverse function on $\mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \mathcal{D}_{z,\epsilon}$ and using (3.119)-(3.120) one gets the lower bound

$$\left| 1 - e^{2i\pi F^{(\varrho)}} \right| > \frac{3c}{2} \quad \text{on} \quad \mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \mathcal{D}_{z,\epsilon}. \quad (3.121)$$

It remains to focus on $\widehat{F}^{(\varrho)}$. It follows from $d(\partial \mathcal{V}_F, Z^{(s)} - i\zeta) > C'$, equation (3.117), the fact that $|\mathbf{b} - \varrho| = O(L^{-1})$ and the estimate (B.6) on $\widehat{\xi}_{\Upsilon_{\text{sing}}}$ that

$$\|\widehat{F}^{(\mathbf{b})} - \widehat{F}_{\text{reg}}^{(\mathbf{b})}\|_{L^\infty(\partial \mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \partial \mathcal{D}_{z,\epsilon})} = O(L^{-\infty}) \quad (3.122)$$

and thus

$$\|\widehat{F}^{(b)} - F^{(\varrho)}\|_{L^\infty(\partial\mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \partial\mathcal{D}_{z,\epsilon})} = O(L^{-1}) \quad (3.123)$$

owing to (3.52). Thus, for L large enough,

$$\left| e^{2i\pi F^{(\varrho)}} - e^{2i\pi \widehat{F}^{(b)}} \right| < \frac{c}{2} \quad \text{on} \quad \partial\mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \partial\mathcal{D}_{z,\epsilon}. \quad (3.124)$$

The function $1 - e^{2i\pi \widehat{F}^{(b)}}$ is meromorphic in \mathcal{V}_F with simple poles at $\{\beta_a^{(s)}\}_1^{n'_{\text{sg}}} = \{Z^{(s)} - i\zeta\} \cap \mathcal{V}_F$. Applying the meromorphic generalisation of Rouché theorem to the functions $1 - e^{2i\pi F^{(\varrho)}}$ and $1 - e^{2i\pi \widehat{F}^{(b)}}$ on \mathcal{V}_F ensures that $1 - e^{2i\pi \widehat{F}^{(b)}}$ has $n'_{\text{sg}} + |Z_F^{(\infty)}|$ zeroes in \mathcal{V}_F . Then, by focusing on $\mathcal{D}_{z,\epsilon}$ with $z \in Z_F^{(\infty)}$, since $1 - e^{2i\pi \widehat{F}^{(b)}}$ is holomorphic there owing to the lower bound (3.117), one gets that $1 - e^{2i\pi \widehat{F}^{(b)}}$ admits a unique zero $w_a \in \mathcal{D}_{z,\epsilon}$ that is simple. The spacing properties of the $z \in Z_F^{(\infty)}$ then ensure that (3.111) holds. Also, the lower bound (3.119) holds with $2c$ replaced by c and $F_\infty^{(\varrho)}|_{\alpha=0}$ replaced by $\widehat{F}^{(b)}$.

It remains to focus on the neighbourhoods $\mathcal{D}_a \equiv \mathcal{D}_{\beta_a^{(s)}, R|\Im(\beta_a^{(s)})|}$ of the $\beta_a^{(s)}$'s. Here $R > 0$ will be assumed large enough. Let

$$\widehat{F}_a^{(b)} = \widehat{F}_{\text{reg}}^{(b)} - L \widehat{\xi}_{\text{sing}}^{(a)} \quad \text{with} \quad \widehat{\xi}_{\text{sing}}^{(a)}(\omega) = \frac{1}{2i\pi L} \sum_{\substack{\beta \in Z^{(s)} - i\zeta \\ \beta \neq \beta_a^{(s)}}} \ln \left(\frac{\sinh(\beta - \omega)}{\sinh(\beta^* - \omega)} \right). \quad (3.125)$$

Then, by virtue of (3.37), the estimate (B.8) ensures that, on \mathcal{D}_a ,

$$\widehat{F}_a^{(b)} = \widehat{F}_{\text{reg}}^{(b)} + O(L^{-\infty}) \quad \text{leading to} \quad \|\widehat{F}_a^{(b)} - F^{(\varrho)}\|_{L^\infty(\mathcal{D}_a)} \leq \frac{C'}{L}. \quad (3.126)$$

Thus, it follows from (3.121) that one has the lower bound $|1 - e^{2i\pi \widehat{F}_a^{(b)}}| \geq c$ on $\overline{\mathcal{D}_a}$. Since, on $\partial\mathcal{D}_a$,

$$\left| e^{2i\pi \widehat{F}_a^{(b)}} \right| \leq e^{C' n_{\text{tot}}^{(\text{msv})} |\Im(\beta_a^{(s)})| R} \leq C' \quad (3.127)$$

one has the bound

$$\left| e^{2i\pi \widehat{F}^{(b)}} - e^{2i\pi \widehat{F}_a^{(b)}} \right| \leq C \left| \frac{\sinh[\sqrt{(R^2 + 2)} |\Im(\beta_a^{(s)})|]}{\sinh[R |\Im(\beta_a^{(s)})|]} - 1 \right| \leq \frac{c}{2} \quad (3.128)$$

provided that R is large enough.

One is in position to apply the meromorphic generalisation of Rouché's theorem to the functions $1 - e^{2i\pi \widehat{F}_a^{(b)}}$ and $1 - e^{2i\pi \widehat{F}^{(b)}}$. Since $1 - e^{2i\pi \widehat{F}_a^{(b)}}$ is holomorphic on \mathcal{D}_a while $1 - e^{2i\pi \widehat{F}^{(b)}}$ has a simple pole at $\beta_a^{(s)}$, it follows that it has also a simple zero $\beta_a^{(s)}$ on \mathcal{D}_a . The upper bound appearing in the *lhs* of (3.112) follows from the fact that $\beta_a^{(s)} \in \mathcal{D}_a$. Going back to the very definition of $\beta_a^{(s)}$ and using that $|e^{2i\pi \widehat{F}_a^{(b)}}|$ is bounded on \mathcal{D}_a , one gets the upper and lower bound in the *rhs* of (3.112). Adopting the parametrisation $\beta_a^{(s)} = \beta_a^{(s)} + \tau$ one gets from the *rhs* of (3.112)

$$\left| \frac{\beta_a^{(s)} - (\beta_a^{(s)})^*}{\beta_a^{(s)} - \beta_a^{(s)}} \right| = \left| 1 + 2i \frac{\Im(\beta_a^{(s)})}{\tau} \right| \leq C \quad \text{so that} \quad \frac{2|\Im(\beta_a^{(s)})|}{\tau} \leq C + 1. \quad (3.129)$$

It thus only remains to establish the lower bounds (3.114)-(3.114) stated in point *iii*). Clearly there exists $\delta > 0$ defining the contours \mathcal{C} such that $\mathcal{C}^c \mathcal{V}_F \setminus \bigcup_{z \in Z_F^{(\infty)}} \mathcal{D}_{z,\epsilon}$. There, (3.121) implies that, in particular, one has $|1 - e^{2i\pi F^{(\varrho)}}| > 3c/2$ on \mathcal{C} . The bounds (B.7), (3.52) and $|\varrho - b| = O(L^{-1})$ ensure that $\|\widehat{F}^{(b)} - F^{(\varrho)}\|_{W_k(\mathcal{C})} = O(L^{-1})$. On that account, the lower bounds (3.114)-(3.114) follow provided that L large enough. \blacksquare

4 The form factors of local operators

Recall that $|\Upsilon\rangle$ stands for the Eigenvector associated with the set Υ of Bethe roots solving the α_Υ -twisted Bethe Ansatz equation (3.19). In its turn, the vector $|\Lambda^{(\alpha)}\rangle$ stands for the α_Λ -twisted ground state, namely the Bethe vector built out of the α_Λ deformation of the ground state Bethe equations (3.7).

4.1 A regular representation for the form factors

The longitudinal and transverse form factors admit determinant representations [30]. A rewriting of these expressions that is more adapted for the further handlings of this paper was obtained in [27]. *Per se*, this rewriting is only valid if $\Upsilon^{(\text{in})} \cap \Lambda^{(\alpha)} = \emptyset$ as, otherwise, one should understand the formulae as limits of coinciding parameters. Dealing with such limits would introduce various technical complications to the large-volume analysis. One can bypass this problem by deforming the α_Λ -twisted ground state roots $\Lambda^{(\alpha)} = \Lambda_0^{(\alpha)} \hookrightarrow \Lambda_b^{(\alpha)}$, c.f. (3.11), entering in the expression of the form factor so that one has always $\Lambda_b^{(\alpha)} \cap \Upsilon = \emptyset$ provided that b belongs to some close loop \mathcal{L} around the origin. If the deformation of the form factor is holomorphic b belonging to the interior of \mathcal{L} , then one can build on the calculation of residues so as to reconstruct the original expression from an integration along $b \in \mathcal{L}$.

Prior to stating the main result of this sub-section, namely a contour integral based determinant representations for the form factors, I need to introduce a few shorthand notations.

The set functions \mathcal{D} and \mathcal{W} are defined by the double products

$$\mathcal{W}(\Lambda; \Upsilon) = \frac{\prod_{\lambda \in \Lambda} \prod_{\mu \in \Upsilon} \left\{ \sinh(\lambda - \mu - i\zeta) \sinh(\mu - \lambda - i\zeta) \right\}}{\prod_{\lambda, \lambda' \in \Lambda} \sinh(\lambda - \lambda' - i\zeta) \prod_{\mu, \mu' \in \Upsilon} \sinh(\mu - \mu' - i\zeta)} \quad (4.1)$$

and

$$\mathcal{D}(\Lambda; \Upsilon) = \frac{\prod_{\lambda \neq \lambda' \in \Lambda} \sinh(\lambda - \lambda') \prod_{\mu \neq \mu' \in \Upsilon} \sinh(\mu - \mu')}{\prod_{\lambda \in \Lambda} \prod_{\mu \in \Upsilon} \left\{ \sinh(\lambda - \mu) \sinh(\mu - \lambda) \right\}}. \quad (4.2)$$

Ξ_Ω corresponds to a set-dependent matrix associated with a set of Bethe roots Ω whose entries are given by

$$[\Xi_\Omega]_{ab} = \delta_{ab} + \frac{K(\nu_a - \nu_b)}{L \widehat{\xi}'_\Omega(\nu_b)} \quad \text{upon taking the parametrisation} \quad \Omega = \{\nu_a\}_1^{|\Omega|}. \quad (4.3)$$

Above, $\widehat{\xi}_\Omega$ is the counting functions associated with the roots Ω . As shown in [35], determinants of these matrices are the main building block of the norm of a Bethe vector.

Finally, I need to introduce the coefficients $\widehat{C}^{(\gamma)}[f](\Lambda; \Upsilon)$ with $\gamma = z$ or $\gamma = +$. The parameter γ distinguishes between the case of form factors associated with longitudinal ($\gamma = z$) or transverse ($\gamma = +$) operators. The longitudinal coefficient takes the form

$$\widehat{C}^{(z)}[f](\Lambda; \Upsilon) = \frac{2}{\pi^2} \sin^2 \left(\frac{1}{2} (\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda} - \pi\alpha) \right) \cdot \frac{\sin^2 [\pi\alpha]}{\sin^2 [\pi f(\theta)]} \cdot \frac{\left(\det_{\Gamma(\Lambda)} [\text{id} + \widehat{\mathbf{u}}_{\alpha; \theta}^{(z)}[f]] \right)^2}{\prod_{\epsilon = \pm} \left\{ V_{\Upsilon; \Lambda}(\theta + \epsilon i\zeta) \right\}}. \quad (4.4)$$

The definition of $\widehat{C}^{(z)}[f](\Lambda; \Upsilon)$ contains an arbitrary parameter θ . The fact that the ratio defining $\widehat{C}^{(z)}[f](\Lambda; \Upsilon)$ does not depend on θ has been established in [24]. In its turn, the transverse coefficient reads

$$\widehat{C}^{(+)}[f](\Lambda; \Upsilon) = \sin^2(\zeta) \cdot \frac{\left(\det_{\Gamma(\Lambda)}[\text{id} + \widehat{U}_{\alpha;\theta}^{(+)}[f]]\right)^2}{\prod_{\epsilon=\pm} \{V_{\Upsilon;\Lambda}(\epsilon i\zeta/2)\}}. \quad (4.5)$$

The expression for $\widehat{C}^{(\gamma)}[f](\Lambda; \Upsilon)$ involves Fredholm determinants of the integral operators $\widehat{U}_{\alpha;\theta}^{(\gamma)}[f]$ acting on $L^2(\Gamma(\Lambda))$, where the contour $\Gamma(\Lambda)$ is a small counterclockwise loop around the set Λ which avoids and does not surround any other singularities of the integral kernel.

I stress that, *a priori*, the expression for $\widehat{C}^{(\gamma)}[f](\Lambda; \Upsilon)$ is only well defined when $\Lambda \cap \Upsilon = \emptyset$, the function f is holomorphic on some small neighbourhood of Λ , $f(\theta) \notin \mathbb{Z}$ and the function $1 - e^{2i\pi f}$ does not vanish on Λ .

The integral kernels of the operators $\widehat{U}_{\alpha;\theta}^{(\gamma)}[f]$ take the form

$$\widehat{U}_{\alpha;\theta}^{(\gamma)}[f](\omega, \omega') = V_{\Upsilon;\Lambda}^{-1}(\omega' + i\zeta) \cdot V_{\Upsilon;\Lambda}(\omega') \cdot \frac{\mathcal{K}_{\alpha;\theta}^{(\gamma)}(\omega, \omega')}{1 - e^{2i\pi f(\omega')}} \quad (4.6)$$

where $V_{\Upsilon;\Lambda}$ has been defined in (3.42),

$$\mathcal{K}_{\alpha;\theta}^{(z)}(\omega, \omega') = K_{\alpha}(\omega - \omega') - K_{\alpha}(\theta - \omega') \quad \text{with} \quad K_{\alpha}(\omega) = \frac{1}{2i\pi} \left\{ e^{2i\pi\alpha} \coth(\omega - i\zeta) - \coth(\omega + i\zeta) \right\} \quad (4.7)$$

and

$$\mathcal{K}_{\alpha;\theta}^{(+)}(\omega, \omega') = \frac{1}{2i\pi} \left\{ \frac{\sinh(\omega' + 3i\zeta/2)}{\sinh(\omega' - i\zeta/2) \sinh(\omega - \omega' - i\zeta)} - \frac{e^{2i\pi\alpha} \sinh(\omega' - 3i\zeta/2)}{\sinh(\omega' + i\zeta/2) \sinh(\omega - \omega' + i\zeta)} \right\}. \quad (4.8)$$

Note that only the longitudinal kernel $\mathcal{K}_{\alpha;\theta}^{(z)}$ does exhibit a dependence on θ .

Proposition 4.1. *The form factors of local operators admit the representation*

$$\frac{\langle \Lambda^{(\alpha)} | \sigma_1^z | \Upsilon \rangle \langle \Upsilon | \sigma_{m+1}^z | \Lambda^{(\alpha)} \rangle}{\langle \Lambda^{(\alpha)} | \Lambda^{(\alpha)} \rangle \cdot \langle \Upsilon | \Upsilon \rangle} \Big|_{\alpha_{\Lambda} = \alpha_{\Upsilon}} = e^{im\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda^{(\alpha)}}} \cdot \oint_{\substack{\partial \mathcal{D}_{0,r_L} \\ \setminus \{\pm r_L\}}} \frac{\partial^2}{\partial \alpha^2} \widehat{\mathcal{S}}^{(z)}(\Lambda_b^{(\alpha)}; \Upsilon)_{|\alpha=0} \frac{db}{2i\pi b}, \quad (4.9)$$

$$\frac{\langle \Lambda^{(\alpha)} | \sigma_1^- | \Upsilon \rangle \langle \Upsilon | \sigma_{m+1}^+ | \Lambda^{(\alpha)} \rangle}{\langle \Lambda^{(\alpha)} | \Lambda^{(\alpha)} \rangle \cdot \langle \Upsilon | \Upsilon \rangle} \Big|_{\alpha_{\Lambda} = \alpha_{\Upsilon}} = (-1)^m \cdot e^{im\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda^{(\alpha)}}} \cdot \oint_{\substack{\partial \mathcal{D}_{0,r_L} \\ \setminus \{\pm r_L\}}} \widehat{\mathcal{S}}^{(+)}(\Lambda_b^{(\alpha)}; \Upsilon)_{|\alpha=0} \frac{db}{2i\pi b}, \quad (4.10)$$

where $\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda^{(\alpha)}}$ has been defined in (3.65) and $\alpha = \alpha_{\Upsilon} - \alpha_{\Lambda}$. Further, I have set

$$\widehat{\mathcal{S}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon) = \prod_{\lambda \in \Lambda_b^{(\alpha)}} \left\{ \frac{(e^{2i\pi \widehat{F}^{(b)}(\lambda)} - 1) \cdot (e^{-2i\pi \widehat{F}^{(b)}(\lambda)} - 1)}{2i\pi L \widehat{\xi}'_{\Lambda^{(\alpha)}}(\lambda)} \right\} \cdot \frac{\mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \mathcal{W}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \widehat{C}^{(\gamma)}[\widehat{F}^{(b)}](\Lambda_b^{(\alpha)}; \Upsilon)}{\prod_{\mu \in \Upsilon} \{2i\pi L \widehat{\xi}'_{\Upsilon}(\mu)\} \cdot \det[\Xi_{\Upsilon}] \cdot \det[\Xi_{\Lambda_b^{(\alpha)}}]}. \quad (4.11)$$

Finally, the integration in (4.9)-(4.10) runs through a disc of radius $r_L = r + \delta r/L$ when r is such that the conclusions of Lemma 3.8 hold while δr as given in Proposition 3.2.

Here, according to (4.3), one has $(\Xi_{\Lambda_b^{(\alpha)}})_{ab} = \delta_{ab} + K(\lambda_a^{(\alpha)}(b) - \lambda_a^{(\alpha)}(b)) \cdot \{L \widehat{\xi}'_{\Lambda^{(\alpha)}}(\lambda_a^{(\alpha)}(b))\}^{-1}$.

Proof—

The starting point is the determinant representation obtained in [27]. It represents the form factors of the local operators as in (4.9)-(4.10) but with the functions $\widehat{\mathcal{S}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon)$ being replaced by

$$\widehat{\mathcal{S}}_{\text{BA}}^{(\gamma)}(\Lambda^{(\alpha)}; \Upsilon) = \prod_{\lambda \in \Lambda^{(\alpha)}} \left\{ \frac{(e^{-2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)} - 1) \cdot (e^{2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)} - 1)}{2i\pi L \widehat{\xi}'_{\Lambda^{(\alpha)}}(\lambda)} \right\} \cdot \frac{\mathcal{D}(\Lambda^{(\alpha)}; \Upsilon) \cdot \mathcal{W}(\Lambda^{(\alpha)}; \Upsilon) \cdot \widehat{C}_{\text{BA}}^{(\gamma)}[\widehat{F}](\Lambda^{(\alpha)}; \Upsilon)}{\prod_{\mu \in \Upsilon} \{2i\pi L \widehat{\xi}'_{\Upsilon}(\mu)\} \cdot \det[\Xi_{\Upsilon}] \cdot \det[\Xi_{\Lambda^{(\alpha)}}]} . \quad (4.12)$$

Since, in such a situation, the integrand does not depend on b , the contour integral can be taken and simply gives 1.

The coefficient $\widehat{C}_{\text{BA}}^{(\gamma)}$ is as defined in (4.4)-(4.5) with the sole difference that one should replace the Fredholm determinant arising in its definition by the determinant of the finite matrix

$$\det_{|\Lambda|} [\text{id} + \widehat{U}_b^{(\gamma)}]_{|b=0} \quad \text{with} \quad (\widehat{U}_b^{(\gamma)})_{k\ell} = \frac{V_{\Upsilon; \Lambda_b^{(\alpha)}}^{-1}(\lambda_{\ell}^{(\alpha)}(b) + i\zeta)}{(V_{\Upsilon; \Lambda_b^{(\alpha)}}^{-1})'(\lambda_{\ell}^{(\alpha)}(b))} \cdot \frac{\mathcal{K}_{\alpha; \theta}^{(\gamma)}(\lambda_k^{(\alpha)}(b), \lambda_{\ell}^{(\alpha)}(b))}{1 - e^{-2i\pi L \widehat{\xi}_{\Upsilon}(\lambda_{\ell}^{(\alpha)}(b))}} . \quad (4.13)$$

The main issue is that the expression (4.12) has an apparent 0/0 indeterminacy if $\Upsilon^{(\text{in})} \cap \Lambda^{(\alpha)} = \emptyset$, which, however, can be resolved. The more general case of $\widehat{\mathcal{S}}_{\text{BA}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon)$ which reduces to (4.13) when $b = 0$. Lemma 7.4 ensures that $\det[\Xi_{\Lambda_b^{(\alpha)}}] \neq 0$ for any $|b| \leq 1$, thus no singularity can issue from the norm determinant. Furthermore, since $\widehat{\xi}_{\Lambda^{(\alpha)}}$ is a biholomorphism on $\mathcal{S}_{\delta}(\mathbb{R})$, $\{\widehat{\xi}'_{\Lambda^{(\alpha)}}(\Lambda_b^{(\alpha)})\} \cap \{0\} = \emptyset$ for any $|b| \leq 1$, provided that L is large enough.

There are three possible origins of poles:

- i) If $\Upsilon \cap \Lambda_b^{(\alpha)} = \Omega \neq \emptyset$ then $1 - e^{-2i\pi L \widehat{\xi}_{\Upsilon}(\lambda_{\ell}^{(\alpha)}(b))} = 0$ if $\lambda_{\ell}^{(\alpha)}(b) \in \Omega$. First assume that it is a first order zero. Then, the pole appearing in the lines ℓ such that $\lambda_{\ell}^{(\alpha)}(b) \in \Omega$, is compensated by the zero appearing in $\{(V_{\Upsilon; \Lambda_b^{(\alpha)}}^{-1})'(\lambda_{\ell}^{(\alpha)}(b))\}^{-1}$. In their turn, the double poles appearing in $\mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon)$ are cancelled by the double zeroes of the prefactors $e^{\pm 2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)} - 1$. Finally, if $\lambda_{\ell}^{(\alpha)}(b) \in \Omega$ is a higher order zero, say $1 - \exp\{-2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)\} = (\lambda - \lambda_{\ell}^{(\alpha)}(b))^k g(\lambda)$, then it is enough to distribute the factors going to zero partly so as to compensate the double poles appearing in $\mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon)$ and partly inside of the lines of the determinant that diverge.
- ii) If $\Upsilon^{(h)} \cap \Lambda_b^{(\alpha)} = \Omega^{(h)} \neq \emptyset$, then the pole appearing in the lines ℓ such that $\lambda_{\ell}^{(\alpha)}(b) \in \Omega^{(h)}$ will be cancelled by the zeroes of the prefactors $e^{\pm 2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)} - 1$.
- iii) The expression is regular when some of the elements of $\Lambda_b^{(\alpha)}$ coincide since the poles appearing in the concerned lines of $\widehat{U}_b^{(\gamma)}$ owing to the vanishing of $(V_{\Upsilon; \Lambda_b^{(\alpha)}}^{-1})'(\lambda_{\ell}^{(\alpha)}(b))$ are compensated by the associated zeroes of the Vandermonde's determinants present in $\mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon)$.

The above pieces of information then allow one to convince oneself that the function

$$G : b \mapsto G(b) \equiv \prod_{\lambda \in \Lambda_b^{(\alpha)}} \left\{ \frac{(e^{-2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)} - 1) \cdot (e^{2i\pi L \widehat{\xi}_{\Upsilon}(\lambda)} - 1)}{2i\pi L \widehat{\xi}'_{\Lambda^{(\alpha)}}(\lambda)} \right\} \cdot \frac{\mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \mathcal{W}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \widehat{C}_{\text{BA}}^{(\gamma)}[\widehat{F}^{(b)}](\Lambda_b^{(\alpha)}; \Upsilon)}{\prod_{\mu \in \Upsilon} \{2i\pi L \widehat{\xi}'_{\Upsilon}(\mu)\} \cdot \det[\Xi_{\Upsilon}] \cdot \det[\Xi_{\Lambda_b^{(\alpha)}}]}$$

is holomorphic on an open neighbourhood of \mathcal{D}_{0,r_L} , with r_L as in the statement of the Proposition. Hence

$$\widehat{\mathcal{S}}_{\text{BA}}^{(\gamma)}(\Lambda^{(\alpha)}; \Upsilon) = G(0) = \oint_{\partial \mathcal{D}_{0,r_L}} G(\mathfrak{b}) \cdot \frac{d\mathfrak{b}}{2i\pi \mathfrak{b}} = \oint_{\partial \mathcal{D}_{0,r_L} \setminus \{\pm r_L\}} G(\mathfrak{b}) \cdot \frac{d\mathfrak{b}}{2i\pi \mathfrak{b}}, \quad (4.14)$$

where the last equality follows from neglecting the zero measure set $\{\pm r_L\}$.

For any $\mathfrak{b} \in \{\partial \mathcal{D}_{0,r_L} \setminus \{\pm r_L\}\}$ one has $\Lambda_{\mathfrak{b}}^{(\alpha)} \cap \{\mu_a^{(s)}\}_1^{n_{\text{sg}}} \cup \Upsilon^{(\text{in})} = \emptyset$ owing to Proposition 3.2. It then remains to observe that $e^{\mp 2i\pi L \widehat{\xi}_\Upsilon(\lambda)} = e^{\pm 2i\pi \widehat{F}(\mathfrak{b})}$ for any $\lambda \in \Lambda_{\mathfrak{b}}^{(\alpha)}$ and that, by using residue calculations, the discrete determinant can be recast in terms of a Fredholm determinant acting on the contour $\Gamma(\Lambda_{\mathfrak{b}}^{(\alpha)})$, this precisely because the other singularities of the integral kernel are disjoint from $\Lambda_{\mathfrak{b}}^{(\alpha)}$, as stipulated in Lemma 3.8. \blacksquare

4.2 The large-volume behaviour of the form factors

I have now introduced enough notation to state, in detail, the main result of the paper which was already stated, in a weakened form and without making the building blocks explicit, in the introduction. The large- L behaviour is described by a certain amount of auxiliary functions that I first need to discuss. Almost all such functions involve the thermodynamic limit of the shift function F as given in (3.97) and of the centred Fermi boundary Umklapp integers ℓ_v^χ introduced in (3.99). I first focus on the non-universal, *i.e.* operator dependent, part.

To start with, consider the function

$$G(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\chi\} \mid \tau) = \frac{\prod_{p=1}^{p_{\text{max}}} \prod_{a=1}^{n_p^{(c)}} \Phi_{1,p}(c_a^{(p)} - \tau)}{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \Phi_{1,1}(\mu - \tau)} \cdot \prod_{v \in \{L,R\}} \left(\frac{\sinh(\sigma_v q - \tau)}{\sinh(\sigma_v q - \tau - i\zeta)} \right)^{\ell_v^\chi} \quad (4.15)$$

and the integral transform $\mathcal{C}_{I_q}[f](\omega) = \int_{I_q} \coth(s - \omega) f(s) \cdot ds / (2i\pi)$ which arise as the building block of the integral kernel

$$U_{\alpha;\theta}^{(\gamma)}[f](\omega, \tau) = e^{2i\pi[\mathcal{C}_{I_q}[f](\tau) - \mathcal{C}_{I_q}[f](\tau + i\zeta)]} \cdot G(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\chi\} \mid \tau) \cdot \frac{\mathcal{K}_{\alpha;\theta}^{(\gamma)}(\omega, \tau)}{1 - e^{2i\pi f(\tau)}}. \quad (4.16)$$

Let $\varrho \in \partial \mathcal{D}_{0,r}$ with r such that the conclusion of Lemme 3.8 hold. The integral kernel $U_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}](\omega, \tau)$ defines a trace class[†] operator $\mathcal{U}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]$ on $L^2(\Gamma_q)$. Here Γ_q is a small counter-clockwise loop around I_q which avoids the zeroes of $1 - \exp\{2i\pi F^{(\varrho)}(\tau)\}$ but encircles $\pm q$ as well as the elements of the set $\Upsilon_{\text{off}}^{(h)}$. The very existence of the contour is guaranteed by Lemma 3.8.

The Fredholm determinants of the operators $\mathcal{U}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]$ arise as the main building blocks of the non-

[†] The trace class follows from the fact that the integral kernel is smooth and that the operator acts on functions supported on a finite number of compact curves, *c.f.* [11]

universal, operator dependent, part of the form factor's asymptotics

$$C^{(z)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{G}; \{\ell_v^\alpha\} | \varrho) = \frac{2}{\pi^2} \cdot \sin^2\left(\frac{1}{2}(\mathcal{P}_{\text{ex}}^{(\varrho)} - \pi\alpha)\right) \cdot \left(\frac{\sin[\alpha\pi]}{\sin[\pi F^{(\varrho)}(\theta)]}\right)^2 \cdot \left(\det[\text{id} + \mathfrak{U}_{\alpha;\theta}^{(z)}[F^{(\varrho)}]]\right)^2 \\ \times \prod_{\epsilon=\pm} \left\{ \prod_{v \in \{L,R\}} \left\{ \sinh[\theta + i\epsilon\zeta - \sigma_v q] \right\}^{-\ell_v^\alpha} \frac{e^{-2i\pi C_{I_q}[F^{(\varrho)}](\theta+i\epsilon\zeta)} \prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \sinh(\theta + i\epsilon\zeta - \mu)}{\prod_{p=1}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{k=1}^p \sinh\left(\theta + i\epsilon\zeta - c_a^{(p)} - i\frac{\zeta}{2}(p+1-2k)\right)} \right\} \quad (4.17)$$

and

$$C^{(+)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{G}; \{\ell_v^\alpha\} | \varrho) = \sin^2(\zeta) \cdot \left(\det[\text{id} + \mathfrak{U}_{\alpha;\theta}^{(+)}[F^{(\varrho)}]]\right)^2 \\ \times \prod_{\epsilon=\pm} \left\{ \frac{e^{-2i\pi C_{I_q}[F^{(\varrho)}](i\epsilon\zeta/2)} \prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \sinh(i\epsilon\zeta/2 - \mu)}{\prod_{p=1}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{k=1}^p \sinh\left(i\frac{\zeta}{2}(2k + \epsilon - 1 - p) - c_a^{(p)}\right)} \prod_{v \in \{L,R\}} \left\{ \sinh[i\epsilon\zeta/2 - \sigma_v q] \right\}^{-\ell_v^\alpha} \right\}. \quad (4.18)$$

The function $\mathcal{P}_{\text{ex}}^{(\varrho)}$ appearing above is as defined in (3.105).

The next building block of the form factor's asymptotics is the function $\mathfrak{X}_{\text{tot}}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{G}; \{\ell_v^\alpha\} | \varrho)$. Its definition involves the auxiliary integral transforms

$$\mathfrak{N}^{(r)}[f](\mu) = \int_{-q}^q f(s) \sum_{\epsilon=\pm} \left\{ \coth[\mu - s + i\epsilon\frac{\zeta}{2}(r+1)] - \coth[\mu - s + i\epsilon\frac{\zeta}{2}(r-1)] \right\} \cdot ds \quad (4.19)$$

and

$$\mathfrak{N}^{(\text{bd})}[f](\mu) = 2 \int_{-q}^q \frac{f(s) - f(\mu)}{\tanh(s - \mu)} \cdot ds - \sum_{\epsilon=\pm} \int_{-q}^q f(s) \coth(s - \mu + i\epsilon\zeta) \cdot ds. \quad (4.20)$$

The function itself takes the form

$$\mathfrak{X}_{\text{tot}}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{G}; \{\ell_v^\alpha\} | \varrho) = \left| \frac{\sinh(2q)}{\sinh(2q + i\zeta)} \right|^{2\ell_R^\alpha \ell_L^\alpha} \cdot \frac{\mathfrak{X}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{G}; \{\ell_v^\alpha\})}{[\sin(\zeta)]^{n_{\text{off}}^{(p)} + n_{\text{off}}^{(h)}}} \cdot \frac{\prod_{r=1}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \exp\{\mathfrak{N}^{(r)}[F^{(\varrho)}](c_a^{(r)})\}}{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \exp\{\mathfrak{N}^{(1)}[F^{(\varrho)}](\mu)\}} \\ \times \prod_{v \in \{L,R\}} \left\{ \frac{e^{\ell_v^\alpha \mathfrak{N}^{(\text{bd})}[F^{(\varrho)}](\sigma_v q)}}{(2\pi)^{\ell_v^\alpha}} \left(\frac{\sinh(2q)}{\sin(\zeta)} \right)^{(\ell_v^\alpha)^2} \right\} \quad (4.21)$$

where, $\aleph_-^{(1)}$ corresponds to the $-$ boundary value of the function and I agree upon

$$\begin{aligned} \aleph(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\}) &= \prod_{r=2}^{p_{\max}} \left\{ \frac{1}{\sin(r\zeta)} \right\}^{n_r^{(z)}} \cdot \prod_{v \in \{L, R\}} \frac{\prod_{r=1}^{p_{\max}} \prod_{b=1}^{n_r^{(z)}} \left\{ \Phi_{1,r}(\sigma_v q - c_b^{(r)}) \Phi_{1,r}(c_b^{(r)} - \sigma_v q) \right\}^{\ell_v^\times}}{\prod_{\lambda \in \Upsilon_{\text{off}}^{(h)}} \left\{ \Phi_{1,p}(\sigma_v q - \lambda) \Phi_{1,p}(\lambda - \sigma_v q) \right\}^{\ell_v^\times}} \\ &\quad \times \frac{\prod_{p,r=1}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{b=1}^{n_r^{(z)}} \Phi_{r,p}(c_a^{(p)} - c_b^{(r)}) \cdot \prod_{\substack{\lambda \neq \lambda' \\ \lambda, \lambda' \in \Upsilon_{\text{off}}^{(h)}}} \Phi_{1,1}(\lambda - \lambda')}{\prod_{\lambda \in \Upsilon_{\text{off}}^{(h)}} \prod_{p=1}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \Phi_{1,p}(\lambda - c_a^{(p)}) \Phi_{1,p}(c_a^{(p)} - \lambda)} . \quad (4.22) \end{aligned}$$

Finally, I introduce the two functionals

$$\mathsf{H}_0[f] = \int_{-q}^q \frac{f'(s)f(t) - f(s)f'(t)}{2 \tanh(s-t)} dt ds + \sum_{v \in \{L, R\}} \sigma_v f(\sigma_v q) \int_{-q}^q \frac{f(s) - f(\sigma_v q)}{\tanh(s - \sigma_v q)} \cdot ds \quad (4.23)$$

and

$$\mathsf{H}_1[f] = - \int_{-q}^q \frac{f(t)f(s)}{\sinh^2(t-s-i\zeta)} \cdot ds dt . \quad (4.24)$$

Theorem 4.2. Assume that the α_Λ -twisted Bethe roots Υ for an excited state satisfy Hypothesis 3.3. There exist a radius of integration $r \in]0; 1/4[$, fixed by $\Upsilon_{\text{off}}^{(h)}$, \mathfrak{C} and ℓ_v^\times , and such that, uniformly in $|\alpha_\Lambda| \leq L^{-3}$, one has the large- L asymptotic expansion

$$\begin{aligned} \left| \frac{\langle \Upsilon | \sigma_1^\gamma | \Lambda^{(\alpha)} \rangle}{\| \Lambda^{(\alpha)} \| \cdot \| \Upsilon \|} \right|_{\alpha=0}^2 &= \oint_{\partial \mathcal{D}_{0,r}} \prod_{v \in \{L, R\}} \left\{ \frac{G^2(1 - \sigma_v \mathfrak{F}_v^{(\varrho)})}{G^2(1 - \sigma_v [\mathfrak{F}_v^{(\varrho)} - \sigma_v \ell_v])} \frac{\mathcal{R}_{n_p^v, n_h^v}(\{p_a^v\}; \{h_a^v\} | -\sigma_v \mathfrak{F}_v^{(\varrho)})}{(L/2\pi)^{(\mathfrak{F}_v^{(\varrho)} - \sigma_v \ell_v)^2}} \right\} \\ &\quad \times \frac{\mathcal{F}^{(\gamma)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} | \varrho)}{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \{Lp'(\mu)\} \cdot \prod_{r=1}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \{Lp'_r(c_a^{(r)})\}} \cdot \left\{ 1 + \Re_L \right\} \frac{d\varrho}{2i\pi\varrho} . \end{aligned}$$

$\mathfrak{F}_v^{(\varrho)}$ is a constant built from the value of the shift function on the endpoints of the Fermi zone

$$\mathfrak{F}_v^{(\varrho)} = \kappa_v - F^{(\varrho)}(\sigma_v q) \quad (4.25)$$

and $\mathcal{F}^{(\gamma)}$ represents the form factor density squared associated with the massive modes:

$$\begin{aligned} \mathcal{F}^{(\gamma)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} | \varrho) &= (-1)^{|\Lambda| - |\Upsilon^{(\text{in})}| + |\Upsilon_{\text{off}}^{(h)}|} \cdot e^{(\mathsf{H}_0 + \mathsf{H}_1)[F^{(\varrho)}]} \cdot \frac{(\aleph_{\text{tot}} \cdot \tilde{\mathcal{C}}^{(\gamma)})(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} | \varrho)}{\det^2[\text{id} + K]} \\ &\quad \times \prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \left(1 - e^{-2i\pi F^{(\varrho)}(\mu)} \right)^2 \cdot \prod_{v \in \{L, R\}} \left\{ \frac{G^2(1 - \sigma_v [\mathfrak{F}_v^{(\varrho)} - \sigma_v \ell_v])}{(\sinh(2q)p'(q))^{(\mathfrak{F}_v^{(\varrho)} - \sigma_v \ell_v)^2}} \cdot (2\pi)^{-\sigma_v F^{(\varrho)}(\sigma_v q)} \right\} \quad (4.26) \end{aligned}$$

where G is the Barnes function,

$$\widetilde{C}^{(+)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} \mid \varrho) = C^{(+)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} \mid \varrho)|_{\alpha=0}$$

and

$$\widetilde{C}^{(z)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} \mid \varrho) = \partial_\alpha^2 C^{(z)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\times\} \mid \varrho)|_{\alpha=0}.$$

$\mathcal{R}_{n_p^{(p)}, n_h^{(h)}}(\{p_a\}; \{h_a\} \mid \nu)$ represents the form factor density squared associated with the massless modes

$$\begin{aligned} \mathcal{R}_{n_p, n_h}(\{p_a\}; \{h_a\} \mid \nu) = & \left(\frac{\sin(\pi\nu)}{\pi} \right)^{2n_h} \frac{\prod_{a < b}^{n_h} (h_a - h_b)^2 \cdot \prod_{a < b}^{n_p} (p_a - p_b)^2}{\prod_{a=1}^{n_h} \prod_{b=1}^{n_p} (h_a + p_b + 1)^2} \\ & \times \prod_{a=1}^{n_p} \frac{\Gamma^2(1 + p_a + \nu)}{\Gamma^2(1 + p_a)} \cdot \prod_{a=1}^{n_h} \frac{\Gamma^2(1 + h_a - \nu)}{\Gamma^2(1 + h_a)}. \end{aligned} \quad (4.27)$$

Finally, \mathfrak{R}_L is the remainder term that is controlled as

$$\mathfrak{R}_L = O\left(\frac{\ln L + \mathfrak{V} + \mathfrak{V}_{\ln}}{L}\right). \quad (4.28)$$

\mathfrak{V} appearing in the control on the remainder has been introduced in (3.98) given the collection of particle/hole integers parametrising the massless modes associated with the excited state Bethe roots Υ , one has

$$\mathfrak{V}_{\ln} = \sum_{x \in \{p_a^\nu, h_a^\nu\}} (x + 1/2) \left| \ln \left(\frac{x + 1/2}{L} \right) \right|. \quad (4.29)$$

Note that the asymptotic expansion provided by the above theorem does hold, to the very same order of the remainder, if one replaces F with F_∞ in all expressions.

The theorem follows from the integral representation provided by Proposition 4.1 above and from the large- L asymptotics obtained in Section 5 Propositions 5.5 and 5.6, Section 6 Proposition 6.1 and Section 7, Proposition 7.1. One should also note that in order to write the result in the stated form relatively to the integration contour, one should represent the integration variable $\delta\varrho$ in $\mathfrak{b} = \varrho + \delta\varrho/L \in \mathcal{D}_{0, r_L} \setminus \{\pm r_L\}$ as given in Proposition 4.1, as $\delta\varrho = \delta r \cdot \varrho/r$.

4.3 A rewriting appropriate for the thermodynamic limit

As shown in Proposition 4.1, in order to access to the large- L asymptotics of the form factors, it is enough to obtain the asymptotic expansion of $\widehat{\mathcal{S}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon)$, c.f. (4.11), uniformly in $\mathfrak{b} \in \partial\mathcal{D}_{0, r_L} \setminus \{\pm r_L\}$. For this purpose, one should first recast $\widehat{\mathcal{S}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon)$ in a form more suited for taking the large- L limit. After some algebra, one gets the decomposition

$$\widehat{\mathcal{S}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon) = \widehat{\mathcal{D}}_{\text{bk}}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) \cdot \widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \widehat{\mathcal{A}}_{\text{reg}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \widehat{\mathcal{A}}_{\text{sing}}(\Upsilon). \quad (4.30)$$

The factor

$$\widehat{\mathcal{D}}_{\text{bk}}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) = \prod_{\lambda \in \Lambda_b^{(\alpha)}} \left\{ \frac{e^{2i\pi \widehat{F}^{(\text{b})}(\lambda)} - 1}{2i\pi L \widehat{\xi}'_{\Lambda^{(\alpha)}}(\lambda)} \right\} \cdot \prod_{\mu \in \Upsilon^{(\text{in})}} \left\{ \frac{e^{-2i\pi \widehat{F}^{(\text{b})}(\mu)} - 1}{2i\pi L \widehat{\xi}'_{\Upsilon}(\mu)} \right\} \cdot \mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) \quad (4.31)$$

gathers the contribution of the roots contained inside of \mathcal{C} . The coefficient

$$\begin{aligned} \widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon) &= (-1)^{|\Upsilon^{(h)}|} (-i)^{|\Upsilon^{(z)}|} \cdot \mathcal{D}(\Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) \cdot \mathcal{W}(\Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) \\ &\times \prod_{p=2}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{k=2}^p \left\{ -i \sinh(\delta_{a,k-1}^{(p)} - \delta_{a,k}^{(p)}) \right\} \cdot \frac{\prod_{\mu \in \Upsilon_{\text{tot}}^{(z)}} V^2(\mu)}{\prod_{\mu \in \Upsilon^{(h)}} V_{\mu}^2(\mu)} \cdot \prod_{\epsilon=\pm 1} \left\{ \frac{\prod_{\mu \in \Upsilon^{(h)}} V(\mu + i\epsilon\zeta)}{\prod_{\mu \in \Upsilon_{\text{tot}}^{(z)}} V(\mu + i\epsilon\zeta)} \right\} \cdot \frac{\prod_{\mu \in \Upsilon^{(h)}} \{2i\pi L_{\xi}^{\widehat{\Upsilon}}(\mu)\}}{\prod_{\mu \in \Upsilon^{(p)}} \{2i\pi L_{\xi}^{\widehat{\Upsilon}}(\mu)\}} \end{aligned}$$

takes into account all the "regular" prefactors depending on the particle and complex valued and roots

$$\Upsilon_{\text{tot}}^{(z)} = \Upsilon^{(p)} \cup \Upsilon^{(z)} \quad (4.32)$$

just as on the hole roots $\Upsilon^{(h)}$. This factor does not contain any exponentially large or small behaviour in L . $\widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon)$ is defined in terms of the auxiliary functions

$$V(\omega) \equiv V_{\Upsilon^{(\text{in})}; \Lambda_b^{(\alpha)}}(\omega) = \frac{\prod_{\mu \in \Upsilon^{(\text{in})}} \sinh(\omega - \mu)}{\prod_{\lambda \in \Lambda_b^{(\alpha)}} \sinh(\omega - \lambda)} \quad \text{and} \quad V_{\nu}(\omega) = \frac{\prod_{\substack{\mu \in \Upsilon^{(\text{in})} \\ \mu \neq \nu}} \sinh(\omega - \mu)}{\prod_{\lambda \in \Lambda_b^{(\alpha)}} \sinh(\omega - \lambda)} \quad (4.33)$$

for any $\nu \in \Upsilon^{(\text{in})}$. The coefficient $\widehat{\mathcal{A}}_{\text{reg}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon)$ contains the operator-dependent contributions to the form factor:

$$\widehat{\mathcal{A}}_{\text{reg}}^{(\gamma)}(\Lambda_b^{(\alpha)}; \Upsilon) = \mathcal{W}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) \cdot \frac{\widehat{C}^{(\gamma)}[\widehat{F}^{(b)}](\Lambda_b^{(\alpha)}; \Upsilon)}{\det[\Xi_{\Lambda_b^{(\alpha)}}] \cdot \det[\Xi_{\Upsilon^{(\text{in})}}]} \cdot \frac{\prod_{\lambda \in \Lambda_b^{(\alpha)}} (e^{-2i\pi \widehat{F}^{(b)}(\lambda)} - 1)}{\prod_{\mu \in \Upsilon^{(\text{in})}} (e^{-2i\pi \widehat{F}^{(b)}(\mu)} - 1)}. \quad (4.34)$$

Finally, $\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon)$ contains the different terms present in (4.11) that, when taken individually, generate an exponentially large or small contributions in L :

$$\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon) = \prod_{\mu \in \Upsilon^{(z)}} \left\{ \frac{1}{2\pi L_{\xi}^{\widehat{\Upsilon}}(\mu)} \right\} \prod_{p=2}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{k=2}^p \left\{ \frac{1}{-i \sinh(\delta_{a,k-1}^{(p)} - \delta_{a,k}^{(p)})} \right\} \cdot \frac{\det[\Xi_{\Upsilon^{(\text{in})}}]}{\det[\Xi_{\Upsilon}]}. \quad (4.35)$$

5 Analysis of $\widehat{\mathcal{D}}_{\text{bk}}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})})$ and $\widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon)$

5.1 Integral representations at finite L

The $i\pi$ -periodic Cauchy transform subordinate to the contour \mathcal{C} refers to the below integral transform

$$\mathbb{C}_{\mathcal{C}}[f](\omega) = \int_{\mathcal{C}} f(s) \coth(s - \omega) \cdot \frac{ds}{2i\pi}. \quad (5.1)$$

$\mathbb{C}_{\mathcal{C}}$ plays an important role in the analysis to come. Its \pm -boundary values on \mathcal{C} will be denoted by $\mathbb{C}_{\mathcal{C}; \pm}[f]$. Given f piecewise continuous on \mathcal{C} and continuous on $\mathcal{C}^{(\pm)} = \mathcal{C} \cap \mathbb{H}^{\pm}$, the Cauchy transforms allow one to define two auxiliary transforms

$$\mathcal{L}_{\mathcal{C}}[f](\omega) = -\mathbb{C}_{\mathcal{C}}[f](\omega) + \mathfrak{d}[f](\widehat{q}_R) \ln[\sinh(\omega - \widehat{q}_R)] - \mathfrak{d}[f](\widehat{q}_L) \ln[\sinh(\omega - \widehat{q}_L)] \quad (5.2)$$

and

$$\widetilde{\mathcal{L}}_{\mathcal{C}}[f](\omega) = -C_{\mathcal{C}}[f](\omega) + \mathfrak{d}[f](\widehat{q}_R) \ln [\sinh (\widehat{q}_R - \omega)] - \mathfrak{d}[f](\widehat{q}_L) \ln [\sinh (\widehat{q}_L - \omega)] \quad (5.3)$$

where the jump operator $\mathfrak{d}[f]$ is as defined in (3.59). As already stated earlier on, \ln appearing above refers to the principal branch of the logarithm. Note that one has to recourse to \pm -boundary values to define the $\mathcal{L}_{\mathcal{C}}$ transforms on \mathbb{R} . The \pm boundary values of these transforms on \mathcal{C} will be denoted as $\mathcal{L}_{\mathcal{C};\pm}$ and $\widetilde{\mathcal{L}}_{\mathcal{C};\pm}$.

Finally, the double integral transform

$$\mathcal{A}_{\eta}[f, g] = \int_{\mathcal{C}} \frac{ds}{2i\pi} \int_{\mathcal{C}' \subset \mathcal{C}} \frac{dt}{2i\pi} \frac{f'(t) g(s)}{\tanh(t - s - i\eta)} \quad (5.4)$$

will also be of use at some later stage. Here, the contour \mathcal{C}' is a contour contained in \mathcal{C} but infinitesimally close to it so that the poles at $t = s + i\eta$ are located outside of \mathcal{C}' . Note that this prescription is only necessary if $\eta = 0^+$.

Lemma 5.1. *Let $\chi \in \{\Lambda_b^{(\alpha)}, \Upsilon\}$ and let $\eta \in \mathbb{R}$ be generic and small enough. Define the sums*

$$f_{\eta}(\omega | \chi) = \sum_{\alpha \in \chi^{(\text{in})}} \ln [\sinh (\omega - \alpha - i\eta)] \quad \text{and} \quad \widetilde{f}_{\eta}(\omega | \chi) = \sum_{\alpha \in \chi^{(\text{in})}} \ln [\sinh (\alpha - \omega - i\eta)] \quad (5.5)$$

where

$$\chi^{(\text{in})} = \Lambda_b^{(\alpha)} \quad \text{if} \quad \chi = \Lambda_b^{(\alpha)} \quad \text{and} \quad \chi^{(\text{in})} = \Upsilon^{(\text{in})} \quad \text{if} \quad \chi = \Upsilon. \quad (5.6)$$

Then, the function f_{η} and \widetilde{f}_{η} can be recast as

$$f_{\eta}(\omega | \chi) = \mathcal{L}_{\mathcal{C}}[\widehat{u}_{\chi}](\omega - i\eta) + \mathbf{1}_{\text{Int}(\mathcal{C})}(\omega - i\eta) \cdot \widehat{u}_{\chi}(\omega - i\eta) + \delta_{\chi; \Upsilon} \sum_{\alpha \in \Upsilon} \ln [\sinh (\omega - \alpha - i\eta)] \quad (5.7)$$

$$\widetilde{f}_{\eta}(\omega | \chi) = \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{u}_{\chi}](\omega + i\eta) + \mathbf{1}_{\text{Int}(\mathcal{C})}(\omega + i\eta) \cdot \widehat{u}_{\chi}(\omega + i\eta) + \delta_{\chi; \Upsilon} \sum_{\alpha \in \Upsilon} \ln [\sinh (\alpha - \omega - i\eta)] \quad (5.8)$$

where $\delta_{\chi; \Upsilon}$ is as defined in (3.75). Also, the arguments of the indicator functions $\text{Int}(\mathcal{C})$ should be understood in $\mathbb{C}/\{i\pi\mathbb{Z}\}$.

Likewise, given $\Omega, \chi \in \{\Lambda_b^{(\alpha)}, \Upsilon\}$ the double sum

$$\mathcal{S}_{\eta}(\Omega; \chi) = \sum_{\alpha \in \Omega^{(\text{in})}} \sum_{\beta \in \chi^{(\text{in})}} \ln [\sinh (\alpha - \beta - i\eta)] \quad (5.9)$$

can be re-expressed in the form

$$\begin{aligned} \mathcal{S}_{\eta}(\Omega; \chi) &= \mathcal{A}_{\eta}[\widehat{u}_{\Omega}, \widehat{u}_{\chi}] + \sum_{\alpha \in \Omega^{(\text{in})}} \mathbf{1}_{\text{Int}(\mathcal{C})}(\alpha - i\eta) \cdot \widehat{u}_{\chi}(\alpha - i\eta) + \delta_{\Omega; \Upsilon} \sum_{\alpha \in \Upsilon} \mathcal{L}_{\mathcal{C}}[\widehat{u}_{\chi}](\alpha - i\eta) \\ &+ \sum_{v \in \{L, R\}} \sigma_v \mathfrak{d}[\widehat{u}_{\chi}](\widehat{q}_v) \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{u}_{\Omega}](\widehat{q}_v + i\eta) + \delta_{\chi; \Upsilon} \sum_{\alpha \in \Upsilon} \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{u}_{\Omega}](\alpha + i\eta) \\ &+ \delta_{\chi; \Upsilon} \sum_{\alpha \in \Upsilon} \mathbf{1}_{\text{Int}(\mathcal{C})}(\alpha + i\eta) \cdot \widehat{u}_{\Omega}(\alpha + i\eta) - \delta_{\Omega; \Upsilon} \delta_{\chi; \Upsilon} \ln \mathcal{W}_{\eta}(\{\beta_a^{(s)}\}_1^{n_{\text{sg}}}; \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}). \end{aligned} \quad (5.10)$$

Above, it is understood that $\widehat{q}_{L/R} + i\eta \in \text{Ext}(\mathcal{C})/\{i\pi\mathbb{Z}\}$. Also, given two sets A, B , $\ln \mathcal{W}_\eta$ is defined as

$$\begin{aligned} \ln \mathcal{W}_\eta(A; B) = & \sum_{\substack{\lambda \in A \\ \mu \in B}} \left\{ \ln [\sinh (\lambda - \mu - i\eta)] + \ln [\sinh (\mu - \lambda - i\eta)] \right\} \\ & - \sum_{\lambda, \lambda' \in A} \ln [\sinh (\lambda - \lambda' - i\eta)] - \sum_{\mu, \mu' \in B} \ln [\sinh (\mu - \mu' - i\eta)] . \end{aligned} \quad (5.11)$$

Proof —

By definition of the set $\chi^{(\text{in})}$, one gets that

$$f_\eta(\omega | \chi) = \int_{\mathcal{C} \setminus \{\omega - i\eta\}} \widehat{u}'_\chi(s) \ln [\sinh (\omega - s - i\eta)] \cdot \frac{ds}{2i\pi} + \delta_{\chi; \Upsilon} \sum_{\alpha \in \Upsilon} \ln [\sinh (\omega - \alpha - i\eta)] . \quad (5.12)$$

Above, $\mathcal{C} \setminus \{\omega - i\eta\}$ corresponds to the contour \mathcal{C} where one has removed a tiny loop around the line $\omega - i\eta + \mathbb{R}^+$ so as to avoid the cut of the logarithm. Note that this regularisation of the contour \mathcal{C} is only necessary when $\omega - i\eta$ lies inside of \mathcal{C} . The second term is only present when $\chi = \Upsilon$ owing to (3.53). The claim then results upon an integration by parts followed by a straightening of the contour up to \mathcal{C} and picking up the pole at $s = \omega - i\eta$.

The handlings are similar in what concerns the re-writing of $\widetilde{f}_\eta(\omega | \chi)$. Finally, by using the expression for $f_\eta(\omega | \chi)$ one recasts $\mathcal{S}_\eta(\Omega; \chi)$ as

$$\begin{aligned} \mathcal{S}_\eta(\Omega; \chi) = & \mathcal{A}_\eta[\widehat{u}_\Omega, \widehat{u}_\chi] + \mathfrak{d}[\widehat{u}_\chi](\widehat{q}_R) \widetilde{f}_\eta(\widehat{q}_R | \Omega) - \mathfrak{d}[\widehat{u}_\chi](\widehat{q}_L) \widetilde{f}_\eta(\widehat{q}_L | \Omega) \\ & + \sum_{\alpha \in \Upsilon} \left\{ \delta_{\chi; \Upsilon} \cdot \widetilde{f}_\eta(\alpha | \Omega) - \delta_{\Omega; \Upsilon} \cdot C_{\mathcal{C}}[\widehat{u}_\chi](\alpha - i\eta) \right\} + \sum_{\alpha \in \Omega^{(\text{in})}} \mathbf{1}_{\text{Int}(\mathcal{C})}(\alpha - i\eta) \cdot \widehat{u}_\chi(\alpha - i\eta) . \end{aligned} \quad (5.13)$$

Upon inserting the obtained expression for \widetilde{f}_η and after some algebra, one gets the claim. \blacksquare

The above lemma allows one to recast various simple and double products appearing in the expression for the form factors. In fact, all the double and single products arising in the intermediate expressions will be recast as certain integral transformations of the function

$$\widehat{z}(\omega) = \widehat{u}_\Upsilon(\omega) - \widehat{u}_{\Lambda_b^{(v)}}(\omega) \quad \text{which has jumps} \quad \mathfrak{d}[\widehat{z}](\widehat{q}_v) = -\kappa_v \quad \text{for } v \in \{L, R\} . \quad (5.14)$$

To start with, I provide expressions for the function $V_{\Upsilon; \Lambda_b^{(v)}}(\omega)$ defined in (3.42).

Corollary 5.2. *Let ω be such that, either, $\omega \in \text{Ext}(\mathcal{C})/\{i\pi\mathbb{Z}\}$ or $\omega \in \text{Int}(\mathcal{C})/\{i\pi\mathbb{Z}\}$ and $e^{2i\pi \widehat{F}^{(b)}(\omega)} = 1$. Then, it holds*

$$V_{\Upsilon; \Lambda_b^{(v)}}(\omega) = e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\omega)} \prod_{\mu \in \Upsilon_{\text{tot}}^{(z)} \setminus \Upsilon^{(h)}} \sinh(\omega - \mu) \cdot \prod_{\alpha \in \Upsilon} \sinh(\omega - \alpha) \quad (5.15)$$

and $\Upsilon_{\text{tot}}^{(z)}$ is as defined in (4.32).

Furthermore, any such ω that is also uniformly away from $\pm q$ and satisfies $d(\omega, \Upsilon^{(z)}) > cL^{-\kappa}$, one has the large- L expansion

$$V_{\Upsilon; \Lambda_b^{(v)}}(\omega) = e^{2i\pi C_{\mathcal{L}}[F^{(v)}](\omega)} \cdot \frac{\prod_{r=1}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \prod_{k=1}^r \sinh(\omega - c_a^{(r)} - i\frac{\zeta}{2}(r+1-2k))}{\prod_{v \in \{L, R\}} \{\sinh(\omega - \sigma_v q)\}^{-\ell_v^\kappa} \cdot \prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \sinh(\omega - \mu)} \cdot \left(1 + \mathcal{O}\left(\frac{\mathfrak{g}}{L}\right)\right) \quad (5.16)$$

where ℓ_v^k is as defined in (3.99) and $F^{(\omega)}$ as defined in (3.50). The expansion holds, to the same degree of precision in the remainder, with $F^{(\omega)}$ replaced by $F_\infty^{(\omega)}$ defined in (3.97). Finally, I_q^\uparrow is a small deformation of I_q which avoids ω from above, in the case when $\omega \in \mathcal{C}$.

Prior to stating the re-writing of \mathcal{D}_{bk} and \mathcal{W} , one should introduce a convenient parametrisations of the local behaviour of the counting function $\widehat{\xi}_\Upsilon$ in a neighbourhood of $\beta_a^{(s)}$:

$$e^{-2i\pi\widehat{\xi}_\Upsilon(\omega)} = e^{-2i\pi\widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega)} \cdot \frac{\sinh[\omega - (\beta_a^{(s)})^*]}{\sinh[\omega - \beta_a^{(s)}]}. \quad (5.17)$$

The function $\widehat{\xi}_{\Upsilon_{\text{reg}}}^{(a)}$ is called the locally regular counting function. Provided that hypothesis (3.37) holds and L is large enough, $\exp\{-2i\pi\widehat{\xi}_{\Upsilon_{\text{reg}}}(\omega)\}$ is non-vanishing for any ω such that $|\omega - \beta_a^{(s)}| \leq |\Im(\beta_a^{(s)})|^{1/4}$.

Proposition 5.3. *One has the integral representations*

$$\widehat{\mathcal{D}}_{\text{bk}}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) = (-1)^{|\Lambda| - |\Upsilon^{(\text{in})}|} \exp\left\{\mathcal{A}_0[\widehat{z}, \widehat{z}] + \sum_{\nu \in \{L, R\}} \sigma_\nu \mathfrak{d}[\widehat{z}](\widehat{q}_\nu) \widetilde{\mathcal{L}}_{\mathcal{C}; -}[\widehat{z}](\widehat{q}_\nu)\right\} \cdot \mathcal{R}_{\text{bk}} \quad (5.18)$$

where $\sigma_R = +$, $\sigma_L = -$, and, recalling (5.17),

$$\mathcal{R}_{\text{bk}} = \mathcal{D}(\{\beta_a^{(s)}\}_1^{n_{\text{sg}}}; \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}) \cdot \prod_{\alpha \in \overline{1}} \left\{ \frac{e^{\widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\alpha) + \mathcal{L}_{\mathcal{C}}[\widehat{z}](\alpha)}}{1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}(\alpha)}} \right\} \cdot \prod_{a=1}^{n_{\text{sg}}} \left\{ \frac{-e^{-2i\pi\widehat{\xi}_{\Upsilon_{\text{reg}}}^{(a)}(\beta_a^{(s)})}}{2i\pi\widehat{\xi}_{\Upsilon}^{(a)}(\mu_a^{(s)})} \cdot \sinh[\beta_a^{(s)} - (\beta_a^{(s)})^*] \right\}. \quad (5.19)$$

Likewise, it holds

$$\mathcal{W}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) = \exp\left\{-\mathcal{A}_\zeta[\widehat{z}, \widehat{z}] - \mathfrak{d}[\widehat{z}](\widehat{q}_R) \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\widehat{q}_R + i\zeta) + \mathfrak{d}[\widehat{z}](\widehat{q}_L) \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\widehat{q}_L + i\zeta)\right\} \cdot \mathcal{R}_{\mathcal{W}} \quad (5.20)$$

where

$$\mathcal{R}_{\mathcal{W}} = \mathcal{W}(\{\beta_a^{(s)}\}_1^{n_{\text{sg}}}; \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}) \cdot \prod_{\alpha \in \overline{1}} \left\{ e^{-\widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\alpha + i\zeta) - \mathcal{L}_{\mathcal{C}}[\widehat{z}](\alpha - i\zeta)} \right\}. \quad (5.21)$$

Proof—

In order to re-express $\widehat{\mathcal{D}}_{\text{bk}}$, it is convenient to recast the singular product as

$$\mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) = \lim_{\eta \rightarrow 0^+} \left\{ (-i\eta)^{-|\Lambda_b^{(\alpha)}| - |\Upsilon^{(\text{in})}|} e^{\mathcal{S}_\eta(\Lambda_b^{(\alpha)}; \Lambda_b^{(\alpha)}) + \mathcal{S}_\eta(\Upsilon^{(\text{in})}; \Upsilon^{(\text{in})}) - \mathcal{S}_\eta(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) - \mathcal{S}_\eta(\Upsilon^{(\text{in})}; \Lambda_b^{(\alpha)})} \right\}$$

leading to

$$\begin{aligned} \mathcal{D}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})}) &= \lim_{\eta \rightarrow 0^+} \left\{ e^{\mathcal{A}_\eta[\widehat{z}, \widehat{z}] + \mathfrak{d}[\widehat{z}](\widehat{q}_R) \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\widehat{q}_R + i\eta) - \mathfrak{d}[\widehat{z}](\widehat{q}_L) \widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\widehat{q}_L + i\eta)} \right. \\ &\quad \times \left. \prod_{\lambda \in \Lambda_b^{(\alpha)}} \left[\frac{1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\lambda - i\eta)}}{-i\eta(1 - e^{-2i\pi\widehat{\xi}_\Upsilon(\lambda - i\eta)})} \right] \prod_{\mu \in \Upsilon^{(\text{in})}} \left[\frac{1 - e^{-2i\pi\widehat{\xi}_\Upsilon(\mu - i\eta)}}{-i\eta(1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu - i\eta)})} \right] \cdot \mathcal{R}_{\text{bk}}(\eta) \right\}. \quad (5.22) \end{aligned}$$

The function \mathcal{R}_{bk} appearing above reads

$$\mathcal{R}_{\text{bk}}(\eta) = \frac{\prod_{\alpha \in \overline{1}} \left\{ e^{\widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\alpha + i\eta) + \mathcal{L}_{\mathcal{C}}[\widehat{z}](\alpha - i\eta)} \right\}}{\mathcal{W}_\eta(\{\beta_a^{(s)}\}_1^{n_{\text{sg}}}; \{\mu_a^{(s)}\}_1^{n_{\text{sg}}})} \prod_{\alpha \in \overline{1}} \left\{ \frac{1 - e^{-2i\pi\widehat{\xi}_\Upsilon(\alpha + i\eta)}}{1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\alpha + i\eta)}} \right\}. \quad (5.23)$$

One can take the $\eta \rightarrow 0^+$ limit of $\mathcal{R}_{\text{bk}}^{(s)}(\eta)$ by using the decomposition (5.17). More precisely, it holds

$$\lim_{\eta \rightarrow 0^+} \{\mathcal{R}_{\text{bk}}(\eta)\} = \mathcal{D}(\{\beta_a^{(s)}\}_1^{n_{\text{sg}}}; \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}) \prod_{\alpha \in \mathbb{T}} \left\{ \frac{e^{\tilde{\mathcal{L}}_{\mathcal{C}}[\tilde{z}](\alpha) + \mathcal{L}_{\mathcal{C}}[\tilde{z}](\alpha)}}{1 - e^{-2i\pi \widehat{L}_{\mathcal{S}}^{(b)}(\alpha)}} \right\} \\ \times \lim_{\eta \rightarrow 0^+} \prod_{a=1}^{n_{\text{sg}}} \left\{ (i\eta)^2 \cdot \frac{1 - e^{-2i\pi \widehat{L}_{\mathcal{S}}(\beta_a^{(s)} + i\eta)}}{1 - e^{-2i\pi \widehat{L}_{\mathcal{S}}(\mu_a^{(s)} + i\eta)}} \right\} \quad (5.24)$$

and it remains to invoke that

$$1 - e^{-2i\pi \widehat{L}_{\mathcal{S}}(\beta_a^{(s)} + i\eta)} \underset{\eta \rightarrow 0^+}{\sim} -\exp \left\{ -2i\pi \widehat{L}_{\mathcal{S}}^{(a)}(\beta_a^{(s)}) \right\} \cdot \frac{\sinh[\beta_a^{(s)} - (\beta_a^{(s)})^*]}{\sinh(i\eta)}. \quad (5.25)$$

The computation relative to $\mathcal{W}(\Lambda_b^{(\alpha)}; \Upsilon^{(\text{in})})$ is very similar, so that I omit the details. \blacksquare

The next proposition rewrites $\widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon)$ in a form that is suited for taking the thermodynamic limit. Prior to stating the result, it appears convenient to introduce the set function

$$\mathcal{G}(\Upsilon^{(p)} \setminus \Upsilon^{(h)}; \{\{c_a^{(p)}\}_{a=1}^{n_p^{(z)}}\}_{p=2}^{p_{\text{max}}}) = \prod_{p=2}^{p_{\text{max}}} \prod_{a=1}^{n_p^{(z)}} \prod_{\lambda \in \Upsilon^{(p)} \setminus \Upsilon^{(h)}} \left\{ \Phi_{1,p}(\lambda - c_a^{(p)}) \Phi_{1,p}(c_a^{(p)} - \lambda) \right\} \\ \times \prod_{p,r=2}^{p_{\text{max}}} \prod_{a=1}^{n_p^{(z)}} \prod_{b=1}^{n_r^{(z)}} \Phi_{r,p}(c_a^{(p)} - c_b^{(r)}) \cdot \prod_{r=2}^{p_{\text{max}}} \left\{ \frac{1}{\sin(r\zeta)} \right\}^{n_r^{(z)}} \quad (5.26)$$

where $c_a^{(p)}$ are the centres of the strings introduced (3.21).

Proposition 5.4. *Let*

$$\Upsilon_0^{(z)} = \Upsilon_{\text{tot}}^{(z)} \setminus \{\mu_a^{(s)}\}_1^{n_{\text{sg}}} \quad \Upsilon_+^{(z)} = \Upsilon_{\text{tot}}^{(z)} \setminus \{(\beta_a^{(s)} + i\zeta)^*\}_1^{n_{\text{sg}}} \quad \text{and} \quad \Upsilon_-^{(z)} = \Upsilon_{\text{tot}}^{(z)} \setminus \{\beta_a^{(s)} + i\zeta\}_1^{n_{\text{sg}}} \quad (5.27)$$

where $\Upsilon_{\text{tot}}^{(z)}$ is as given in (4.32). Further, for $\epsilon \in \{0, \pm 1\}$, set

$$\mathbb{Y} = \Upsilon_{\text{tot}}^{(z)} \setminus \Upsilon^{(h)} \quad \mathbb{Y}_{\epsilon} = \Upsilon_{\epsilon}^{(z)} \setminus \Upsilon^{(h)} \quad \text{and} \quad \iota_0 = 2, \quad \iota_{\pm 1} = -1. \quad (5.28)$$

Assume that Hypotheses (3.36)-(3.37) on the spacing of the string centres hold. Then, within the convention (1.6), it holds

$$\widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon) = (-1)^{|\Upsilon^{(h)}|} (\mathcal{D} \cdot \mathcal{W})(\Upsilon^{(h)}; \Upsilon^{(p)}) \cdot \mathcal{G}(\Upsilon^{(p)} \setminus \Upsilon^{(h)}; \{\{c_a^{(p)}\}_{a=1}^{n_p^{(z)}}\}_{p=2}^{p_{\text{max}}}) \\ \times \prod_{\mu \in \mathbb{Y}} \prod_{\epsilon=0, \pm 1} \left\{ e^{\iota_{\epsilon} \mathcal{L}_{\mathcal{C}}[\tilde{z}](\mu + i\epsilon\zeta)} \right\} \cdot \frac{\prod_{\mu \in \Upsilon^{(h)}} \left(1 - e^{-2i\pi \widehat{F}^{(b)}(\mu)} \right)^2}{\prod_{\mu \in \Upsilon^{(h)} \cup \Upsilon^{(p)}} \{2i\pi \widehat{L}_{\mathcal{S}}'(\mu)\}} \cdot \mathcal{R}_{\text{ex}} \quad (5.29)$$

where

$$\begin{aligned}
\mathcal{R}_{\text{ex}} = & \prod_{\epsilon=0,\pm 1} \prod_{\mu \in \mathbb{Y}_\epsilon} \prod_{a=1}^{n_{\text{sg}}} \left\{ \frac{\sinh(\mu + i\epsilon\zeta - \beta_a^{(s)})}{\sinh(\mu + i\epsilon\zeta - \mu_a^{(s)})} \right\}^{\epsilon} \\
& \times \prod_{a \neq b}^{n_{\text{sg}}} \left\{ \frac{\sinh^3(\mu_a^{(s)} - \beta_b^{(s)}) \cdot \sinh(\mu_a^{(s)} - (\beta_b^{(s)})^*)}{\sinh^2(\mu_a^{(s)} - \mu_b^{(s)}) \cdot \sinh(\beta_a^{(s)} - \beta_b^{(s)}) \cdot \sinh(\beta_a^{(s)} - (\beta_b^{(s)})^*)} \right\} \\
& \times \prod_{a=1}^{n_{\text{sg}}} \left\{ \frac{\exp\{2i\pi L \widehat{\xi}_{\Gamma_{\text{reg}}^{(a)}}(\beta_a^{(s)})\} (2i\pi L \widehat{\xi}'_{\Gamma}(\mu_a^{(s)}) \sinh(\mu_a^{(s)} - \beta_a^{(s)}))^2}{2i\pi K(\mu_a^{(s)} - \Re(\beta_a^{(s)}) \mid \Im(\beta_a^{(s)})) \sinh(\beta_a^{(s)} - (\beta_a^{(s)})^*)} \right. \\
& \quad \times \left. \frac{(1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}}(\beta_a^{(s)}))(1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}}((\beta_a^{(s)})^*))}{(1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}}(\mu_a^{(s)}))^2} \right\} \left(1 + \mathcal{O}\left(\frac{n_{\text{tot}}^2}{L^\infty}\right)\right). \quad (5.30)
\end{aligned}$$

The $1 + \mathcal{O}(n_{\text{tot}}^2/L^\infty)$ corrections appearing above issue solely from the string deviations. Their expression can be inferred from the content of the proof. Finally, n_{tot} has been defined in (3.22).

Proof—

Upon applying the η -regularisation procedure as in the proof of Proposition 5.3, one gets

$$\prod_{\mu \in \Upsilon_{\text{tot}}^{(z)}} V^2(\mu) = \prod_{\mu \in \Upsilon_{\text{tot}}^{(z)}} e^{2\mathcal{L}_{\mathcal{C}}[\widehat{z}](\mu)} \cdot \prod_{a=1}^{n_{\text{sg}}} \left\{ \frac{2i\pi L \widehat{\xi}'_{\Gamma}(\mu_a^{(s)}) \sinh(\mu_a^{(s)} - \beta_a^{(s)})}{1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}}(\mu_a^{(s)})} \cdot \prod_{\mu \in \Upsilon_0} \frac{\sinh(\mu - \beta_a^{(s)})}{\sinh(\mu - \mu_a^{(s)})} \prod_{a \neq b}^{n_{\text{sg}}} \frac{\sinh(\mu_a^{(s)} - \beta_b^{(s)})}{\sinh(\mu_a^{(s)} - \mu_b^{(s)})} \right\}^2$$

and

$$\prod_{\mu \in \Upsilon^{(h)}} V_\mu^2(\mu) = \prod_{\mu \in \Upsilon^{(h)}} \left\{ e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\mu)} \cdot \frac{2i\pi L \widehat{\xi}'_{\Gamma}(\mu)}{1 - e^{-2i\pi F^{(b)}}(\mu)} \right\}^2 \cdot \prod_{a=1}^{n_{\text{sg}}} \prod_{\mu \in \Upsilon^{(h)}} \left(\frac{\sinh(\mu - \beta_a^{(s)})}{\sinh(\mu - \mu_a^{(s)})} \right)^2 \quad (5.31)$$

Further, one has

$$\begin{aligned}
\prod_{\mu \in \Upsilon_{\text{tot}}^{(z)}} \prod_{\epsilon=\pm 1} V^2(\mu + i\epsilon\zeta) = & \prod_{\mu \in \Upsilon_{\text{tot}}^{(z)}} \prod_{\epsilon=\pm 1} e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\mu + i\epsilon\zeta)} \cdot \prod_{a=1}^{n_{\text{sg}}} \left\{ \frac{e^{2i\pi L \widehat{\xi}_{\Gamma_{\text{reg}}^{(a)}}(\beta_a^{(s)})} \sinh^2(\beta_a^{(s)} - (\beta_a^{(s)})^*)}{\sinh(\mu_a^{(s)} - \beta_a^{(s)}) \sinh(\mu_a^{(s)} - (\beta_a^{(s)})^*)} \right\} \\
& \times \prod_{a \neq b}^{n_{\text{sg}}} \left\{ \frac{\sinh(\beta_a^{(s)} - \beta_b^{(s)}) \cdot \sinh(\beta_a^{(s)} - (\beta_b^{(s)})^*)}{\sinh(\mu_a^{(s)} - \beta_b^{(s)}) \cdot \sinh(\mu_a^{(s)} - (\beta_b^{(s)})^*)} \right\} \cdot \frac{\prod_{\epsilon=\pm 1} \prod_{\mu \in \Upsilon_\epsilon^{(z)}} \prod_{\alpha \in \Gamma} \sinh(\mu - \alpha + i\epsilon\zeta)}{\prod_{a=1}^{n_{\text{sg}}} \left\{ (1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}}(\beta_a^{(s)}))(1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(a)}}^{(b)}}((\beta_a^{(s)})^*)) \right\}}. \quad (5.32)
\end{aligned}$$

Finally, it holds

$$\prod_{\mu \in \Upsilon^{(h)}} \prod_{\epsilon=\pm 1} V^2(\mu + i\epsilon\zeta) = \prod_{\mu \in \Upsilon^{(h)}} \prod_{\epsilon=\pm 1} e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\mu + i\epsilon\zeta)} \prod_{\epsilon=\pm 1} \prod_{\mu \in \Upsilon^{(h)}} \prod_{\alpha \in \Gamma} \sinh(\mu - \alpha + i\epsilon\zeta). \quad (5.33)$$

The obtained expressions can be slightly simplified by using that

$$2i\pi K(\mu_a^{(s)} - \Re(\beta_a^{(s)}) \mid \Im(\beta_a^{(s)})) = \frac{\sinh[\beta_a^{(s)} - (\beta_a^{(s)})^*]}{\sinh[\mu_a^{(s)} - (\beta_a^{(s)})^*] \cdot \sinh[\mu_a^{(s)} - \beta_a^{(s)}]}. \quad (5.34)$$

It remains to recast the double products as

$$(\mathcal{D} \cdot \mathcal{W})(\Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) = \mathcal{P}^{(1)}(\Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) \cdot \mathcal{P}^{(2)}(\Upsilon^{(h)}) \cdot \mathcal{P}^{(2)}(\Upsilon_{\text{tot}}^{(z)}) \quad (5.35)$$

where

$$\mathcal{P}^{(1)}(A; B) = \prod_{\substack{\lambda \in A \\ \mu \in B}} \left\{ \frac{\sinh(\lambda - \mu - i\zeta) \sinh(\mu - \lambda - i\zeta)}{\sinh(\lambda - \mu) \sinh(\mu - \lambda)} \right\} \quad \text{and} \quad \mathcal{P}^{(2)}(A) = \frac{\prod_{\lambda \neq \lambda' \in A} \sinh(\lambda - \lambda')}{\prod_{\substack{\lambda, \lambda' \\ \in A}} \sinh(\lambda - \lambda' - i\zeta)}. \quad (5.36)$$

For further convenience, it is useful to parametrise $\Upsilon_{\text{tot}}^{(z)}$ defined in (4.32) in terms of the string centres $\tilde{c}_a^{(r)}$ as

$$\Upsilon_{\text{tot}}^{(z)} = \left\{ \left\{ \tilde{c}_a^{(r)} + i\frac{\zeta}{2}(r+1-2k) + \delta_{a,k}^{(r)} \right\}_{k=1}^r \right\}_{a=1}^{\tilde{n}_z^{(r)}} \Bigg\}_{r=1}^{p_{\max}}. \quad (5.37)$$

Here, the sole difference between $\tilde{c}_a^{(p)}$ and the string centres $c_a^{(p)}$ introduced in (3.21) and (3.94) is for $p = 1$ where the $\tilde{c}_a^{(p)}$ also include the elements of $\Upsilon^{(p)}$ which squeeze down to $\pm q$, *viz.* the rapidities of the roots $\Upsilon_v^{(p)}$. In particular, one has $\tilde{n}_z^{(p)} = n_p^{(z)}$ for $p \geq 2$ and $\tilde{n}_z^{(1)} = |\Upsilon^{(p)}|$.

By using the spacing hypothesis (3.36) on central element of odd strings, one readily gets that

$$\mathcal{P}^{(1)}(\Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) = \prod_{\lambda \in \Upsilon^{(h)}} \prod_{p=1}^{p_{\max}} \prod_{a=1}^{\tilde{n}_z^{(p)}} \frac{1}{\Phi_{p,1}(\lambda - \tilde{c}_a^{(p)} - \delta_{a,\lfloor \frac{p+1}{2} \rfloor}^{(p)}) \cdot \Phi_{1,p}(\tilde{c}_a^{(p)} + \delta_{a,\lfloor \frac{p+1}{2} \rfloor}^{(p)} - \lambda)} \cdot \left(1 + O\left(\frac{n_{\text{tot}}^2}{L^\infty}\right)\right) \quad (5.38)$$

where $\Phi_{r,p}$ is given by (2.8) and the $O(n_{\text{tot}}^2/L^\infty)$ remainder issue from neglecting the string deviations. One can drop, on the level of the obtained formula, the remaining string deviation. Quite similarly, one has

$$\begin{aligned} \mathcal{P}^{(2)}(\Upsilon_{\text{tot}}^{(z)}) &= \prod_{\substack{p,r=1 \\ (p,a) \neq (r,b)}}^{p_{\max}} \prod_{a=1}^{\tilde{n}_z^{(p)}} \prod_{b=1}^{\tilde{n}_z^{(r)}} \Phi_{r,p}(\tilde{c}_a^{(p)} - \tilde{c}_b^{(r)}) \cdot \prod_{p=2}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \frac{\prod_{k \neq s}^p \sinh[i\zeta(s-k)]}{\prod_{k,s=1}^p \sinh[\delta_{a,k}^{(p)} - \delta_{a,s}^{(p)} + i\zeta(s-k-1)]} \\ &\quad \times \left\{ \frac{1}{\sinh(-i\zeta)} \right\}^{\tilde{n}_z^{(1)}} \cdot \left(1 + O\left(\frac{n_{\text{tot}}^2}{L^\infty}\right)\right). \end{aligned} \quad (5.39)$$

In the penultimate product one has to keep the string deviations for terms such that $s = k - 1$ but, otherwise, these produce as well $1 + O(n_{\text{tot}}^2/L^\infty)$ contributions. Eventually, one gets

$$\begin{aligned} \mathcal{P}^{(2)}(\Upsilon_{\text{tot}}^{(z)}) &= \frac{\prod_{\substack{a \neq b \\ a,b=1}}^{\tilde{n}_z^{(1)}} \sinh[\tilde{c}_a^{(1)} - \tilde{c}_b^{(1)}]}{\prod_{a,b=1}^{\tilde{n}_z^{(1)}} \sinh[\tilde{c}_a^{(1)} - \tilde{c}_b^{(1)} - i\zeta]} \cdot \frac{\prod_{\substack{p,r=1 \\ (p,a) \neq (r,b)}}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{b=1}^{n_r^{(z)}} \Phi_{r,p}(\tilde{c}_a^{(p)} - \tilde{c}_b^{(r)})}{\prod_{p=2}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{k=2}^p \sinh[\delta_{a,k-1}^{(p)} - \delta_{a,k}^{(p)}]} \cdot \frac{1 + O(n_{\text{tot}}^2/L^\infty)}{\prod_{p=2}^{p_{\max}} \{(-1)^p \sinh(ip\zeta)\}^{n_p^{(z)}}}. \end{aligned} \quad (5.40)$$

Thus,

$$\begin{aligned}
& (-i)^{|\Upsilon^{(z)}|} (\mathcal{D} \cdot \mathcal{W})(\Upsilon^{(h)}; \Upsilon_{\text{tot}}^{(z)}) \prod_{p=2}^{p_{\max}} \prod_{a=1}^{n_p^{(z)}} \prod_{k=2}^p \{ -i \sinh [\delta_{a,k-1}^{(p)} - \delta_{a,k}^{(p)}] \} \\
& = (\mathcal{D} \cdot \mathcal{W})(\Upsilon^{(h)}; \Upsilon^{(p)}) \cdot \mathcal{G}(\Upsilon^{(p)} \setminus \Upsilon^{(h)}; \{ \{ c_a^{(p)} \}_{a=1}^{n_p^{(z)}} \}_{p=2}^{p_{\max}}) \cdot \left(1 + O\left(\frac{n_{\text{tot}}^2}{L^\infty}\right) \right). \quad (5.41)
\end{aligned}$$

■

5.2 The large- L expansion

Proposition 5.5. *Let $\mathfrak{b} = \varrho + \delta\varrho/L \in \mathcal{D}_{0,r_L}$ with r_L as in Proposition 4.1. Assume that (3.36) and (3.37) holds. Then, one has*

$$\begin{aligned}
& \widehat{\mathcal{D}}_{\text{bk}}(\Lambda_{\mathfrak{b}}^{(\alpha)}; \Upsilon^{(\text{in})}) \cdot \mathcal{R}_{\text{ex}} = (-1)^{|\Lambda_{\mathfrak{b}}^{(\alpha)}| - |\Upsilon^{(\text{in})}| + \kappa_L(\kappa_L - \kappa_R)} [\sinh(2q)]^{(\kappa_L - \kappa_R)^2} e^{H_0[F^{(\varrho)}]} \\
& \times \prod_{v \in \{L, R\}} \left\{ \exp \left[-2\sigma_v \kappa_v \int_{-q}^q \frac{F^{(\varrho)}(s) - F^{(\varrho)}(\sigma_v q)}{\tanh(s - \sigma_v q)} \cdot ds \right] \right. \\
& \quad \times \frac{G(1 - \widehat{\mathfrak{f}}_v^{(\varrho)}, 1 + \widehat{\mathfrak{f}}_v^{(\varrho)}) e^{-i\frac{\pi}{2}\sigma_v (\widehat{F}_{\text{reg}}^{(\mathfrak{b})}(\widehat{q}_v))^2 - i\pi\sigma_v \widehat{\mathfrak{f}}_v^{(\mathfrak{b})} \kappa_v}}{[Lp'(\sigma_v q) \sinh(2q)/2\pi] (\widehat{\mathfrak{f}}_v^{(\varrho)})^2} \left. \right\} \cdot \left(1 + O\left(\frac{\ln L}{L}\right) \right) \quad (5.42)
\end{aligned}$$

where the functional H_0 has been introduced in (4.23), G is the Barnes function, $\widehat{\mathfrak{f}}_v^{(\varrho)}$ is as given by (4.25), \mathcal{R}_{ex} has been defined in (5.30) while

$$\widehat{\mathfrak{f}}_v^{(\mathfrak{b})} = \kappa_v - \widehat{F}_{\text{reg}}^{(\mathfrak{b})}(\widehat{q}_v) \quad (5.43)$$

Note that in order to have a better given control on the final remainder, one has to keep, at this stage, some of the terms still at their finite- L values. These will subsequently compensate with some other terms in the expansion.

Proof—

Starting from the rewriting provided by Proposition 5.3, after invoking equation (B.2) of Lemma B.1 and using the string centres spacing (3.37) assumption, one obtains that $\mathcal{R}_{\text{ex}} \cdot \mathcal{R}_{\text{bk}} = 1 + O(n_{\text{tot}}^2/L^\infty)$. Further, as a direct consequence of Lemma A.4 and of Corollary A.3, one gets that

$$\begin{aligned}
& e^{\mathcal{A}_0[\widehat{z}, \widehat{z}] - \sum_{v \in \{L, R\}} \sigma_v \kappa_v \widehat{\mathcal{L}}_{\mathcal{C}; -}[\widehat{z}](\widehat{q}_v)} = (-1)^{\kappa_L(\kappa_L - \kappa_R)} [\sinh(\widehat{q}_R - \widehat{q}_L)]^{(\kappa_L - \kappa_R)^2} e^{\widehat{H}_{\kappa_L, \kappa_R}[\widehat{F}_{\text{reg}}^{(\mathfrak{b})}]} \\
& \times \prod_{v \in \{L, R\}} \left\{ \frac{G(1 - \widehat{\mathfrak{f}}_v^{(\mathfrak{b})}, 1 + \widehat{\mathfrak{f}}_v^{(\mathfrak{b})}) e^{-i\frac{\pi}{2}\sigma_v (\widehat{F}_{\text{reg}}^{(\mathfrak{b})}(\widehat{q}_v))^2 - i\pi\sigma_v \widehat{\mathfrak{f}}_v^{(\mathfrak{b})} \kappa_v}}{[L\widehat{\xi}'_\Lambda(\widehat{q}_v) \sinh(\widehat{q}_R - \widehat{q}_L)] (\widehat{\mathfrak{f}}_v^{(\mathfrak{b})})^2} \right\} \cdot \left(1 + O\left(\frac{\ln L}{L}\right) \right) \quad (5.44)
\end{aligned}$$

with

$$\widehat{\mathfrak{f}}_v^{(\mathfrak{b})} = \kappa_v - \widehat{F}_{\text{reg}}^{(\mathfrak{b})}(\widehat{q}_v) \quad (5.45)$$

and where the functional $\widehat{\mathbb{H}}_{\kappa_L; \kappa_R}$ takes the explicit form

$$\begin{aligned} \widehat{\mathbb{H}}_{\kappa_L; \kappa_R}[\widehat{F}_{\text{reg}}^{(b)}] &= \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{\widehat{F}'_{\text{reg}}(s) \widehat{F}_{\text{reg}}^{(b)}(t) - \widehat{F}_{\text{reg}}^{(b)}(s) \widehat{F}'_{\text{reg}}(t)}{2 \tanh(s-t)} dt ds \\ &\quad - \sum_{v \in \{L, R\}} \sigma_v(\widehat{\mathbf{f}}_v^{(b)} + \kappa_v) \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{\widehat{F}_{\text{reg}}^{(b)}(s) - \widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v)}{\tanh(s - \widehat{q}_v)} \cdot ds. \end{aligned} \quad (5.46)$$

A straightforward expansion based on (3.6) shows that

$$\begin{aligned} \widehat{\mathbb{H}}_{\kappa_L; \kappa_R}[\widehat{F}_{\text{reg}}^{(b)}] &+ (\kappa_R - \kappa_L)^2 \ln \sinh(\widehat{q}_R - \widehat{q}_L) \\ &= \widehat{\mathbb{H}}_{\kappa_L; \kappa_R}[\widehat{F}_{\text{reg}}^{(b)}]_{\widehat{q}_v \hookrightarrow \sigma_v q} + (\kappa_R - \kappa_L)^2 \ln \sinh(2q) + \mathcal{O}\left(\frac{\|\widehat{F}_{\text{reg}}^{(b)}\|_{L^\infty(\mathcal{S}_\delta(\mathbb{R}))}}{L}\right). \end{aligned}$$

The claim then follows upon recalling the estimate (3.52). \blacksquare

The next proposition utilises the partitioning (3.88) of the set of particle and hole roots into the massless and massive modes and the shorthand notation for the discrepancies between the number of particles and holes in the left, right and off-boundary collections of roots (3.93).

Proposition 5.6. *Let p_a^v, h_a^v be the integers parametrising the particle-hole Bethe roots squeezing on the Fermi zone, c.f. the decomposition (3.88), (3.90) and (3.91). Under the notations and assumptions of Proposition 5.5, it holds*

$$\begin{aligned} \widehat{\mathcal{D}}_{\text{ex}}(\Lambda_b^{(\alpha)}; \Upsilon) \cdot \mathcal{R}_{\text{ex}}^{-1} &= (-1)^{|\Upsilon_{\text{off}}^{(h)}|} \cdot \frac{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} (1 - e^{-2i\pi F^{(\psi)}(\mu)})^2}{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)} \cup \Upsilon_{\text{off}}^{(p)}} \{L p'(\mu)\}} \cdot \mathfrak{X}_{\text{tot}}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^\kappa\}) \cdot \left| \frac{\sinh(2q)}{\sinh(2q + i\zeta)} \right|^{2\kappa_R \kappa_L} \\ &\prod_{v \in \{L, R\}} \left\{ \frac{\mathcal{R}_{n_v^{(p)}, n_v^{(h)}}(\{p_a^v\}; \{h_a^v\} \mid -\sigma_v \mathbf{f}_v^{(\varrho)})}{[L \sinh(2q) p'(\sigma_v q) / 2\pi]^{(\mathbf{f}_v^{(\varrho)} - \sigma_v \ell_v)^2 - (\mathbf{f}_v^{(\varrho)})^2}} \cdot \frac{e^{\sigma_v \kappa_v \mathfrak{N}^{(\text{bd})}[F^{(\varphi)}](\sigma_v q)}}{(2\pi)^{\kappa_v \sigma_v}} \left(\frac{\sin(\zeta)}{\sinh(2q)} \right)^{\kappa_v^2} \right\} \\ &\quad \times \left\{ 1 + \mathcal{O}\left(\frac{\mathfrak{V} + \mathfrak{V}_{\text{ln}} + \ln L}{L}\right) \right\} \end{aligned} \quad (5.47)$$

where $\mathfrak{V}, \mathfrak{V}_{\text{ln}}$ are as defined in (3.92) and (4.29), $\mathcal{R}_{n_p^v, n_h^v}$ is given by (4.27), the integral transform $\mathfrak{N}^{(\text{bd})}$ has been defined in (4.20) while $\mathfrak{X}_{\text{tot}}$ can be found in (5.26) and $\mathbf{f}_v^{(\varrho)}$ has been defined in (4.25).

Proof —

It is straightforward to deduce from the large- L expansion (3.92) of the L or R particle/hole roots that

$$\begin{aligned} \mathcal{G}(\Upsilon^{(p)} \setminus \Upsilon^{(h)}; \{\{c_a^{(r)}\}_{a=1}^{n_r^{(z)}}\}_{r=2}^{p_{\text{max}}}) &= \mathcal{G}(\Upsilon_{\text{off}}^{(p)} \setminus \Upsilon_{\text{off}}^{(h)}; \{\{c_a^{(r)}\}_{a=1}^{n_r^{(z)}}\}_{r=2}^{p_{\text{max}}}) \\ &\times \prod_{r=2}^{p_{\text{max}}} \prod_{a=1}^{n_r^{(z)}} \prod_{v \in \{L, R\}} \left\{ \Phi_{1,r}(\sigma_v q - c_a^{(r)}) \Phi_{1,r}(c_a^{(r)} - \sigma_v q) \right\}^{\ell_v} \cdot \left(1 + \mathcal{O}\left(\frac{\mathfrak{V}}{L}\right) \right). \end{aligned} \quad (5.48)$$

Let $\mathbb{Y}_{\text{off}} = \Upsilon_{\text{off}}^{(z)} \setminus \Upsilon_{\text{off}}^{(h)}$. By virtue of Corollary A.3 and the expansions (3.6), straightforward handlings lead to

$$\prod_{\substack{\mu \in \mathbb{Y}_{\text{off}} \\ \epsilon \in \{\pm 1, 0\}}} e^{\iota_{\epsilon} \mathcal{L}_{\mathcal{C}}[\widehat{z}](\mu + i\epsilon\zeta)} = \frac{\prod_{r=1}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \left\{ e^{\mathfrak{N}^{(r)}[F^{(\varrho)}](c_a^{(r)})} \prod_{v \in \{L, R\}} \left\{ \Phi_{1,r}(\widehat{q}_v - c_a^{(r)}) \Phi_{1,r}(c_a^{(r)} - \widehat{q}_v) \right\}^{-\sigma_v \kappa_v} \right\}}{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \left\{ e^{\mathfrak{N}_{-}^{(1)}[F^{(\varrho)}](\mu)} \prod_{v \in \{L, R\}} \left\{ \Phi_{1,1}(\widehat{q}_v - \mu) \Phi_{1,1}(\mu - \widehat{q}_v) \right\}^{-\sigma_v \kappa_v} \right\}} \cdot \left(1 + \mathcal{O}\left(\frac{1}{L}\right) \right).$$

The integral transforms $\mathfrak{N}^{(r)}$ have been introduced in (4.19) and $\mathfrak{N}_{-}^{(1)}$ corresponds to the $-$ boundary value of the transform.

Corollary A.3, straightforward expansions based on the definition of such roots (3.90)-(3.91), their large- L expansion (3.92) and the form of the uniform expansion of the Gamma function (A.1) lead to

$$\begin{aligned} \prod_{\mu \in \Upsilon_v^{(p)} \setminus \Upsilon_v^{(h)}} \prod_{\epsilon \in \{\pm 1, 0\}} e^{\iota_{\epsilon} \mathcal{L}_{\mathcal{C}}[\widehat{z}](\mu + i\epsilon\zeta)} &= \frac{e^{\ell_v \mathfrak{N}^{(\text{bd})}[F^{(\varrho)}](\sigma_v q)}}{\left\{ L \sinh(2q) \widehat{\xi}'_{\Lambda}(\sigma_v q) \right\}^{-2\sigma_v \ell_v \mathfrak{f}_v^{(\varrho)}}} \prod_{a=1}^{n_v^{(p)}} \Gamma^2 \left(\frac{1 + p_a^v - \sigma_v \mathfrak{f}_v^{(\varrho)}}{1 + p_a^v} \right) \\ &\times \prod_{a=1}^{n_v^{(h)}} \Gamma^2 \left(\frac{1 + h_a^v + \sigma_v \mathfrak{f}_v^{(\varrho)}}{1 + h_a^v} \right) \cdot \prod_{\substack{\mu \in \Upsilon_v^{(p)} \setminus \Upsilon_v^{(h)} \\ \epsilon = \pm}} \left\{ \left(\frac{\sinh(\mu - \widehat{q}_v + i\epsilon\zeta)}{\sinh(\mu - \widehat{q}_{\overline{v}})} \right)^{\sigma_v \kappa_v} \cdot \left(\frac{\sinh(\mu - \widehat{q}_{\overline{v}})}{\sinh(\mu - \widehat{q}_{\overline{v}} + i\epsilon\zeta)} \right)^{\sigma_v \kappa_{\overline{v}}} \right\} \\ &\times \prod_{\mu \in \Upsilon_v^{(h)}} \left\{ e^{2i\pi \widehat{F}_{\text{reg}}^{(\text{b})}(\mu)} \cdot \left\{ 1 + \mathcal{O}\left(\frac{\mathfrak{Y}_{\ln} + \mathfrak{Y} + \ln L}{L}\right) \right\} \right\}. \quad (5.49) \end{aligned}$$

Here, $\overline{v} = L$ if $v = R$ and $\overline{v} = R$ if $v = L$.

The large- L expansion (3.92) of the roots belonging to $\Upsilon_v^{(p/h)}$ and properties *iii*) – *iv*) of Lemma 3.8 ensures that

$$\begin{aligned} \prod_{\mu \in \Upsilon_L^{(h)} \cup \Upsilon_R^{(h)}} \left\{ e^{2i\pi \widehat{F}_{\text{reg}}^{(\text{b})}(\mu)} \right\} \cdot \frac{\prod_{\mu \in \Upsilon^{(h)}} \left(1 - e^{-2i\pi \widehat{F}^{(\text{b})}(\mu)} \right)^2}{(2i\pi)^{|\Upsilon^{(p)}| + |\Upsilon^{(h)}|}} &= \frac{\prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \left(1 - e^{-2i\pi F^{(\varrho)}(\mu)} \right)^2}{(2i\pi)^{n_{\text{off}}^{(p)} + n_{\text{off}}^{(h)}}} \\ &\times \prod_{v \in \{L, R\}} \left\{ \frac{1}{(2i\pi)^{\ell_v}} \cdot \left(\frac{\sin[\pi F^{(\varrho)}(\sigma_v q)]}{\pi} \right)^{2n_v^{(h)}} \right\} \cdot \left(1 + \left(\frac{\mathfrak{Y}}{L} \right) \right). \quad (5.50) \end{aligned}$$

Collecting the κ_v dependent terms issuing from the various $\mathcal{L}_{\mathcal{C}}$ transforms into the function

$$\begin{aligned} \chi(\Upsilon^{(h)}; \Upsilon^{(p)}) &= \prod_{v \in \{L, R\}} \left\{ \prod_{\mu \in \Upsilon_{\text{off}}^{(p)} \setminus \Upsilon_{\text{off}}^{(h)}} \left\{ \Phi_{1,1}(\mu - \widehat{q}_v) \Phi_{1,1}(\widehat{q}_v - \mu) \right\}^{-\sigma_v \kappa_v} \right\} \\ &\times \prod_{v \in \{L, R\}} \prod_{\mu \in \Upsilon_v^{(p)} \setminus \Upsilon_v^{(h)}} \left\{ \prod_{v' \in \{L, R\}} \left\{ \Phi_{1,1}(\mu - \widehat{q}_{\overline{v}}) \Phi_{1,1}(\widehat{q}_{\overline{v}} - \mu) \right\}^{-\sigma_{v'} \kappa_{v'}} \prod_{\epsilon = \pm} \left(\frac{\sinh(\mu - \widehat{q}_R + i\epsilon\zeta)}{\sinh(\mu - \widehat{q}_L + i\epsilon\zeta)} \right)^{\kappa_v} \right\} \quad (5.51) \end{aligned}$$

one has that

$$\begin{aligned}
(\mathcal{WD}_\chi)(\Upsilon^{(h)}; \Upsilon^{(p)}) &= \left\{ \Phi_{1,1}(\widehat{q}_R - \widehat{q}_L) \Phi_{1,1}(\widehat{q}_L - \widehat{q}_R) \right\}^{\kappa_R \kappa_L} \cdot \prod_{v \in \{L, R, \text{off}\}} (\mathcal{WD})(\Upsilon_v^{(h)}; \Upsilon_v^{(p)}) \\
&\times \prod_{\lambda \in \Upsilon_{\text{off}}^\delta} \prod_{\mu \in \widehat{\Upsilon}_L^\delta \cup \widehat{\Upsilon}_R^\delta} \left\{ \Phi_{1,1}(\lambda - \mu) \Phi_{1,1}(\mu - \lambda) \right\} \cdot \prod_{\lambda \in \widehat{\Upsilon}_R^\delta} \prod_{\mu \in \widehat{\Upsilon}_L^\delta} \left\{ \Phi_{1,1}(\lambda - \mu) \Phi_{1,1}(\mu - \lambda) \right\} \\
&\times \prod_{v \in \{L, R\}} \prod_{\mu \in \Upsilon_v^\delta} \left\{ \prod_{\epsilon = \pm} \frac{\sinh(\mu - \widehat{q}_R + i\epsilon\zeta)}{\sinh(\mu - \widehat{q}_L + i\epsilon\zeta)} \Phi_{1,1}^{-\sigma_v}(\mu - \widehat{q}_v) \Phi_{1,1}^{-\sigma_v}(\widehat{q}_v - \mu) \right\}^{\kappa_v}. \quad (5.52)
\end{aligned}$$

I have introduced $\Upsilon_v^\delta = \Upsilon_v^{(p)} \setminus \Upsilon_v^{(h)}$ and $\widehat{\Upsilon}_v^\delta = \Upsilon_v^\delta \cup \{\widehat{q}_v\}^{-\sigma_v \kappa_v}$. There, \widehat{q}_v is repeated $|\kappa_v|$ times and the set should be added if $-\sigma_v \kappa_v \geq 0$ and subtracted otherwise. The large- L expansion of most of the individual terms appearing in the above decomposition can be readily accessed. Some more care is only needed relatively to $(\mathcal{WD})(\Upsilon_v^{(h)}; \Upsilon_v^{(p)})$ when $v \in \{L, R\}$. One should regularise the singular terms as proposed in [25] leading to

$$(\mathcal{WD})(\Upsilon_v^{(h)}; \Upsilon_v^{(p)}) = \frac{(-1)^{\ell_v \frac{\ell_v - 1}{2} - n_v^{(h)}}}{L^{\ell_v^2} \prod_{\lambda \in \Upsilon_v^{(h)} \cup \Upsilon_v^{(p)}} \{\widehat{L\xi'_\Upsilon}(\mu)\}^{-1}} \prod_{\lambda \in \Upsilon_v^\delta} \prod_{\mu \in \Upsilon_v^\delta} \widehat{\Psi}_{1,1}(\lambda, \mu) \frac{\prod_{a < b}^{n_v^{(h)}} (h_a^v - h_b^v)^2 \cdot \prod_{a < b}^{n_v^{(p)}} (p_a^v - p_b^v)^2}{\prod_{a=1}^{n_v^{(h)}} \prod_{b=1}^{n_v^{(p)}} (h_a^v + p_b^v + 1)^2} \quad (5.53)$$

where

$$\widehat{\Psi}_{1,1}(\lambda, \mu) = \frac{\sinh(\lambda - \mu)}{(\widehat{\xi}_\Upsilon(\lambda) - \widehat{\xi}_\Upsilon(\mu)) \sinh(\lambda - \mu - i\zeta)}. \quad (5.54)$$

The asymptotic expansion (3.49) and spacing properties of the singular roots (3.36) and the upper bound (3.39) on the spacing between the singular roots $\alpha_a^{(s)}$ and $\beta_a^{(s)}$ ensure, owing to Lemma B.2, that that for any

$$\lambda \neq \mu \in \Upsilon_v^{(h)} \cup \Upsilon_v^{(p)}, \text{ it holds } \widehat{\xi}_\Upsilon(\lambda) - \widehat{\xi}_\Upsilon(\mu) = (\widehat{\xi}_{\Upsilon_{\text{reg}}}(\lambda) - \widehat{\xi}_{\Upsilon_{\text{reg}}}(\mu))(1 + O(n_{\text{sg}} L^{-\infty})). \quad (5.55)$$

Straightforward expansions then lead to

$$(\mathcal{WD})(\Upsilon_v^{(h)}; \Upsilon_v^{(p)}) = \frac{(-1)^{\ell_v \frac{\ell_v - 1}{2} - n_v^{(h)}} (i)^{\ell_v} \prod_{a < b}^{n_v^{(h)}} (h_a^v - h_b^v)^2 \cdot \prod_{a < b}^{n_v^{(p)}} (p_a^v - p_b^v)^2}{\left\{ \frac{L p'(\sigma_v q)}{(2\pi)} \right\}^{\ell_v^2 - n_v^{(p)} - n_v^{(h)}} \sin^{\ell_v^2}(\zeta) \prod_{a=1}^{n_v^{(h)}} \prod_{b=1}^{n_v^{(p)}} (h_a^v + p_b^v + 1)^2} \cdot \left\{ 1 + O\left(\frac{1 + \mathfrak{U}}{L}\right) \right\}.$$

All-in-all, one gets that

$$\begin{aligned}
(\mathcal{WD}_\chi)(\Upsilon^{(h)}; \Upsilon^{(p)}) &= (\mathcal{WD})(\Upsilon_{\text{off}}^{(h)}; \Upsilon_{\text{off}}^{(p)}) \prod_{\mu \in \Upsilon_{\text{off}}^\delta} \prod_{v \in \{L, R\}} \left\{ \Phi_{1,1}(\mu - \sigma_v q) \Phi_{1,1}(\sigma_v q - \mu) \right\}^{\ell_v^\kappa} \\
&\times \left| \frac{\sinh(2q)}{\sinh(2q - i\zeta)} \right|^{2(\kappa_L \kappa_R + \ell_L^\kappa \ell_R^\kappa)} \prod_{v \in \{L, R\}} \left\{ \left(\frac{\sinh(2q)}{\sinh(\zeta)} \right)^{(\ell_v^\kappa)^2 - \kappa_v^2} \frac{\prod_{a < b}^{n_v^{(h)}} (h_a^v - h_b^v)^2 \cdot \prod_{a < b}^{n_v^{(p)}} (p_a^v - p_b^v)^2}{\prod_{a=1}^{n_v^{(h)}} \prod_{b=1}^{n_v^{(p)}} (h_a^v + p_b^v + 1)^2} \right\} \\
&\times \prod_{v \in \{L, R\}} \left\{ (i)^{\ell_v} (-1)^{n_v^{(h)}} \left(\frac{L p'(\sigma_v q)}{2\pi} \right)^{n_v^{(p)} + n_v^{(h)}} \cdot \left(\frac{2\pi}{L \sinh(2q) p'(\sigma_v q)} \right)^{\ell_v^2} \cdot \left\{ 1 + O\left(\frac{1 + \mathfrak{U}}{L}\right) \right\} \right\}
\end{aligned}$$

Finally, the product $\mathcal{D} \cdot \mathcal{W}$ associated to the off-critical particle-hole excitations can be recast as

$$(\mathcal{D} \cdot \mathcal{W})(\Upsilon_{\text{off}}^{(h)}; \Upsilon_{\text{off}}^{(p)}) = \frac{\prod_{\lambda \neq \lambda' \in \Upsilon_{\text{off}}^{(p)}} \Phi_{1,1}(\lambda - \lambda') \cdot \prod_{\mu \neq \mu' \in \Upsilon_{\text{off}}^{(h)}} \Phi_{1,1}(\mu - \mu')}{\left\{ \sinh(-i\zeta) \right\}^{n_{\text{off}}^{(p)} + n_{\text{off}}^{(h)}} \cdot \prod_{\mu \in \Upsilon_{\text{off}}^{(h)}} \prod_{\lambda \in \Upsilon_{\text{off}}^{(p)}} \left\{ \Phi_{1,1}(\lambda - \mu) \Phi_{1,1}(\mu - \lambda) \right\}}. \quad (5.56)$$

It solely remains to put the expansions of the various terms together. ■

6 Analysis of $\mathcal{A}_{\text{reg}}(\Lambda_{\text{b}}^{(\alpha)}; \Upsilon)$

6.1 Statement of the result

Define the functional

$$\mathcal{W}_{\mathcal{K}_L; \mathcal{K}_R}[f] = \frac{\prod_{\epsilon=\pm} [\sinh(2q + \epsilon i\zeta)]^{\mathcal{K}_R \mathcal{K}_L}}{[\sinh(i\zeta)]^{\mathcal{K}_R^2} [\sinh(-i\zeta)]^{\mathcal{K}_L^2}} \cdot e^{\mathcal{H}_1[f]} \cdot \prod_{\epsilon=\pm} \prod_{v \in \{L, R\}} e^{\sigma_v \mathcal{K}_v \mathcal{C}_{l_q}[2i\pi f](\sigma_v q + \epsilon i\zeta)} \quad (6.1)$$

where the functional \mathcal{H}_1 is given by (4.24).

Proposition 6.1. *Let $\mathfrak{b} = \varrho + \delta\varrho/L \in \mathcal{D}_{0, r_L}$ with r_L as in Proposition 4.1. Assume that the string centre hypothesis holds (3.37), then*

$$\widehat{\mathcal{A}}_{\text{reg}}(\Lambda_{\text{b}}^{(\alpha)}; \Upsilon) = \mathcal{A}_{\text{reg}}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^{\mathcal{K}}\}) \cdot \left\{ 1 + \mathcal{O}\left(\frac{\vartheta}{L}\right) \right\} \quad (6.2)$$

where

$$\begin{aligned} \mathcal{A}_{\text{reg}}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^{\mathcal{K}}\}) &= (-1)^{(\mathcal{K}_R - \mathcal{K}_L)\mathcal{K}_L} \mathcal{W}_{\mathcal{K}_L; \mathcal{K}_R}[F^{(\varrho)}] \cdot \frac{C^{(\gamma)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^{\mathcal{K}}\} \mid \varrho)}{\det^2[\text{id} + K]} \\ &\times \prod_{v \in \{L, R\}} \left\{ (-1)^{\frac{1}{2}\mathcal{K}_v(\mathcal{K}_v - 1)} (2i\pi)^{\sigma_v \mathcal{K}_v} \frac{e^{i\frac{\pi}{2}\sigma_v (F_{\text{reg}}^{(\text{b})}(\widehat{q}_v))^2 + i\pi\sigma_v \mathcal{K}_v \widehat{f}_v^{(\text{b})}}}{(2\pi)^{\sigma_v F^{(\varrho)}(\sigma_v q)}} \cdot \frac{G(1 - \sigma_v \mathcal{F}_v^{(\varrho)})}{G(1 + \sigma_v \mathcal{F}_v^{(\varrho)})} \right\}. \end{aligned} \quad (6.3)$$

where G is the Barnes function and the coefficients $C^{(\gamma)}$ have been introduced in (4.17)-(4.18).

6.2 Analysis

Proposition 6.2. *Assume that the string centre separation hypothesis holds (3.37), then*

$$\mathcal{W}(\Lambda_{\text{b}}^{(\alpha)}; \Upsilon^{(\text{in})}) = (-1)^{(\mathcal{K}_R - \mathcal{K}_L)\mathcal{K}_L} \mathcal{W}_{\mathcal{K}_L; \mathcal{K}_R}[F^{(\varrho)}] \cdot \left(1 + \mathcal{O}\left(\frac{1}{L}\right) \right). \quad (6.4)$$

Proof —

The starting point for the expansion is given by (5.20) appearing in Proposition 5.3. After an integration by parts, the \mathcal{A}_{ζ} transform can be recast as

$$\mathcal{A}_{\zeta}[\widehat{z}, \widehat{z}] = - \sum_{v \in \{L, R\}} \sigma_v \mathfrak{d}[\widehat{z}](\widehat{q}_v) \mathcal{C}_{\mathcal{C}}[\widehat{z}](\widehat{q}_v - i\zeta) + \int_{\mathcal{C}} \frac{\widehat{z}(s) \widehat{z}(t)}{\sinh^2(s - t + i\zeta)} \cdot \frac{ds dt}{(2i\pi)^2}. \quad (6.5)$$

The double integral can be estimated starting from the decomposition

$$\widehat{z}(\omega) = 2i\pi F^{(\varrho)}(\omega) \mathbf{1}_{\mathcal{C}^{(+)}}(\omega) + \widehat{z}(\omega) \quad (6.6)$$

where

$$\widehat{z}(\omega) = \begin{cases} 2i\pi \left(\widehat{F}_{\text{reg}}^{(\text{b})}(\omega) - F^{(\varrho)}(\omega) - \widehat{L}_{\mathcal{S}}^{\text{sing}}(\omega) \right) + \widehat{u}_{\Upsilon}^{(+)}(\omega) - \widehat{u}_{\Lambda_{\text{b}}^{(\alpha)}}^{(+)}(\omega) & \omega \in \mathcal{C}^{(+)} \\ \widehat{u}_{\Upsilon}^{(-)}(\omega) - \widehat{u}_{\Lambda_{\text{b}}^{(\alpha)}}^{(-)}(\omega) & \omega \in \mathcal{C}^{(-)} \end{cases} \quad (6.7)$$

By Lemmas B.2-B.3 and the bound (3.52), one gets that $\|\widehat{z}\|_{L^1(\mathcal{C})} = O(L^{-1})$. The decomposition (6.6) then allows one to recast

$$\begin{aligned} \int_{\mathcal{C}} \frac{\widehat{z}(s)\widehat{z}(t)}{\sinh^2(s-t+i\zeta)} \cdot \frac{dsdt}{(2i\pi)^2} &= \int_{\mathcal{C}^{(+)}} \frac{F^{(\varrho)}(t)F^{(\varrho)}(s)}{\sinh^2(s-t+i\zeta)} dsdt \\ &+ \int_{\mathcal{C}} \frac{ds}{2i\pi} \int_{\mathcal{C}^{(+)}} dt \left\{ \frac{F^{(\varrho)}(t)\widehat{z}(s) + F^{(\varrho)}(s)\widehat{z}(t)}{\sinh^2(s-t+i\zeta)} \right\} + \int_{\mathcal{C}} \frac{dsdt}{(2i\pi)^2} \frac{\widehat{z}(t)\widehat{z}(s)}{\sinh^2(s-t+i\zeta)}. \end{aligned} \quad (6.8)$$

The last two integrals produce $O(L^{-1})$ corrections owing to direct bounds and estimates on the L^1 norm of \widehat{z} and the bound (3.100).

Further, by invoking the large- L asymptotics of the $\mathcal{L}_{\mathcal{C}}$ and $\widetilde{\mathcal{L}}_{\mathcal{C}}$ transforms, the spacing property of the singular roots (3.39) and the string centre spacing hypothesis (3.37) allow one to infer that $\mathcal{R}_{\mathcal{W}} = 1 + O(n_{\text{sg}}^2 \cdot L^{-\infty})$.

Then, by using the large- L behaviour of the Cauchy transforms given in Proposition A.2 and of the bounds (3.52), one gets that

$$\begin{aligned} \mathcal{W}(\Lambda_{\text{b}}^{(\alpha)}; \Upsilon^{(\text{in})}) &= \exp \left\{ - \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{F^{(\varrho)}(t)F^{(\varrho)}(s)}{\sinh^2(s-t+i\zeta)} dsdt + \sum_{\epsilon=\pm} \sum_{v \in \{L,R\}} \sigma_v \mathcal{K}_v \mathcal{C}_{[\widehat{q}_L; \widehat{q}_R]}[2i\pi F^{(\varrho)}](\widehat{q}_v + i\epsilon\zeta) \right. \\ &\quad \left. - \sum_{\substack{v,v' \\ \in \{L,R\}}} \sigma_v \sigma_{v'} \mathcal{K}_v \ln \sinh(\widehat{q}_{v'} - \widehat{q}_v + i\zeta) + O\left(\frac{1}{L}\right) \right\}. \end{aligned} \quad (6.9)$$

It then remains to invoke the expansion of the endpoints (3.6) and then observe that all resulting correction of first order in L^{-1} cancel out, so that, owing to (3.100), the first corrections are a $O\left(\|F^{(\varrho)}\|_{L^\infty(\mathcal{S}_\delta(\mathbb{R}))}^2 \cdot L^{-2}\right)$.

■

I now discuss the large- L asymptotics of the other building blocks of $\widehat{\mathcal{A}}_{\text{reg}}^{(\gamma)}(\Lambda_{\text{b}}^{(\alpha)}; \Upsilon)$. This analysis requires the characterisation of the poles and zeroes of $1 - e^{2i\pi \widehat{F}^{(\text{b})}}$ inside of \mathcal{C} which has been achieved in Lemma 3.8.

Proposition 6.3. *Let $\text{b} = \varrho + \delta\varrho/L \in \mathcal{D}_{0,r_L} \setminus \{\pm r_L\}$ with r_L as in Proposition 4.1 and assume that the string centre separation hypothesis holds (3.37). Then,*

$$\frac{\prod_{\lambda \in \Lambda_{\text{b}}^{(\alpha)}} (e^{-2i\pi \widehat{F}^{(\text{b})}(\lambda)} - 1)}{\prod_{\mu \in \Upsilon^{(\text{in})}} (e^{-2i\pi \widehat{F}^{(\text{b})}(\mu)} - 1)} = \exp \left\{ \oint_{\mathcal{C}} \widehat{z}(s) \ln' [e^{-2i\pi \widehat{F}^{(\text{b})}(s)} - 1] \cdot \frac{ds}{2i\pi} \right\} \cdot \prod_{v \in \{L,R\}} (e^{-2i\pi \widehat{F}^{(\text{b})}(\widehat{q}_v)} - 1)^{\sigma_v \mathcal{K}_v} \cdot \mathfrak{R}_F \quad (6.10)$$

where the remainder takes the form

$$\Re_F = \prod_{a=1}^{n_{\text{sg}}} \frac{1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu_a^{(s)})}}{1 - e^{-2i\pi L \widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\beta_a^{(s)*})}}. \quad (6.11)$$

Furthermore, provided that property (3.12) holds, one has the large- L expansion

$$\frac{\prod_{\lambda \in \Lambda_b^{(\alpha)}} (e^{-2i\pi \widehat{F}^{(b)}(\lambda)} - 1)}{\prod_{\mu \in \Upsilon^{(\text{in})}} (e^{-2i\pi \widehat{F}^{(b)}(\mu)} - 1)} = \prod_{v \in \{L, R\}} \left\{ (-1)^{\frac{1}{2}\kappa_v(\kappa_v-1)} \frac{e^{i\frac{\pi}{2}\sigma_v(\widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v))^2 + i\pi\sigma_v\kappa_v\widehat{F}_v^{(b)}}}{(2i\pi)^{-\sigma_v\kappa_v} \cdot (2\pi)^{\sigma_v F^{(b)}(\sigma_v q)}} \cdot \frac{G(1 - \sigma_v \mathbf{f}_v^{(Q)})}{G(1 + \sigma_v \mathbf{f}_v^{(Q)})} \right\} \cdot \left(1 + O\left(\frac{1}{L}\right)\right) \quad (6.12)$$

with $\mathbf{f}_v^{(Q)}$ given by (4.25).

Proof—

It follows from Lemma 3.8 that, for almost all $\epsilon > 0$, $t \mapsto 1 - e^{-2i\pi \widehat{F}^{(b)}(t+i\epsilon)}$ has no roots on $[\widehat{q}_L; \widehat{q}_R]$. Thus, for these ϵ 's, there exists a small box $V_\epsilon \supset [\widehat{q}_L; \widehat{q}_R]$ passing through \widehat{q}_v such that $t \mapsto 1 - e^{-2i\pi \widehat{F}^{(b)}(t+i\epsilon)}$ has no zeroes on V_ϵ and is holomorphic there. One can thus define on this neighbourhood a holomorphic determination for its logarithm $\ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(t+i\epsilon)} - 1]$. Then, for $\Omega \in \{\Lambda_b^{(\alpha)}, \Upsilon\}$, one gets

$$\begin{aligned} \sum_{\mu \in \Omega^{(\text{in})}} \ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(\mu+i\epsilon)} - 1] &= - \oint_{\partial V_\epsilon} \widehat{u}_\Omega(s) \ln'_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(s+i\epsilon)} - 1] \cdot \frac{ds}{2i\pi} \\ &+ \sum_{v \in \{L, R\}} \sigma_v \mathfrak{d}[\widehat{u}_\Omega](\widehat{q}_v) \ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(\widehat{q}_v+i\epsilon)} - 1] + \delta_{\Omega, \Upsilon} \sum_{\mu \in \Upsilon} \ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(\mu+i\epsilon)} - 1]. \end{aligned} \quad (6.13)$$

Deforming the contour ∂V_ϵ to \mathcal{C} and using that the only singularities of the integrand delimited by these contours are the simple poles of $\ln'_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(s+i\epsilon)} - 1]$ located at

- $Z_\epsilon = \{\mathfrak{z}_a^{(s)} - i\epsilon\}_1^{n_{\text{sg}}}$ with residue 1;
- $B_\epsilon = \{(\beta_a^{(s)})^* - i\epsilon\}_1^{n_{\text{sg}}}$ with residue -1 ;
- $W_\epsilon = \{w_a - i\epsilon\}_{a=1}^{\ell_F}$ with residue 1;

where w_a and $\mathfrak{z}_a^{(s)}$ are the simple zeroes introduced in Lemma 3.8, equation (3.110), one eventually gets

$$\begin{aligned} \sum_{\mu \in \Omega^{(\text{in})}} \ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(\mu+i\epsilon)} - 1] &= - \oint_{\mathcal{C}} \widehat{u}_\Omega(s) \ln'_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(s+i\epsilon)} - 1] \cdot \frac{ds}{2i\pi} \\ &+ \sum_{v \in \{L, R\}} \sigma_v \mathfrak{d}[\widehat{u}_\Omega](\widehat{q}_v) \ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(\widehat{q}_v+i\epsilon)} - 1] + \delta_{\Omega, \Upsilon} \sum_{\mu \in \Upsilon} \ln_\epsilon [e^{-2i\pi \widehat{F}^{(b)}(\mu+i\epsilon)} - 1] + \sum_{\substack{\mu \in Z_\epsilon \\ \cup W_\epsilon \setminus B_\epsilon}} \widehat{u}_\Omega(\mu). \end{aligned} \quad (6.14)$$

Note that, in the ultimate sum that occurs in (6.14), the elements should be summed up according to their respective multiplicities.

The above formula allows one to recast the ϵ -deformed version of the ratio appearing in the *lhs* of (6.10). The representation holds, in fact, for any $\epsilon > 0$ owing to the continuity in ϵ of the original ratio and its rewriting resulting from (6.14). One can then compute the $\epsilon \rightarrow 0^+$ limit to get representation (6.10). In the intermediate calculations, one should use the decomposition (5.17), the fact that $\beta_a^{(s)}$ is a zero of $e^{-2i\pi\widehat{F}^{(b)}}$ while $(\beta_a^{(s)})^*$ is a zero of $e^{-2i\pi\widehat{L}_{\xi^*}^{(b)}}$ and that, for any $z \in Z_0 \cup W_0$, one has

$$\lim_{\epsilon \rightarrow 0^+} \left\{ \frac{1 - e^{-2i\pi\widehat{L}_{\xi^{(b)}}^{(b)}(z-i\epsilon)}}{1 - e^{-2i\pi\widehat{L}_{\xi^*}^{(b)}(z-i\epsilon)}} \right\} = 1. \quad (6.15)$$

The latter limit is a consequence of $e^{-2i\pi\widehat{L}_{\xi^{(b)}}^{(b)}(z)} \neq 1$. Indeed, if this last equation did not hold, then, since $\exp\{-2i\pi\widehat{F}^{(b)}(z)\} = 1$, one would also have that $e^{-2i\pi\widehat{L}_{\xi^*}^{(b)}(z)} = 1$. This would imply that $z \in \{\Upsilon^{(\text{in})} \cup \{\mu_a^{(s)}\}_{a=1}^{n_{\text{sg}}}\} \cap \Lambda_b^{(\alpha)}$. However, the latter set is empty owing to (3.13).

It remains to extract the large- L behaviour. Observe that the remainder can be recast as

$$\Re_F^{-1} = \prod_{a=1}^{n_{\text{sg}}} \left\{ 1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu_a^{(s)})} \frac{e^{-2i\pi[\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}((\beta_a^{(s)})^*) - \widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu_a^{(s)})]} - 1}{1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu_a^{(s)})}} \right\}. \quad (6.16)$$

The property (3.12), the estimate $\|\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}\|_{W_1^\infty(\mathcal{S}_\delta(\mathbb{R}))} < C$ for some $\delta > 0$ and the one on the deviation of $(\beta_a^{(s)})^*$ in respect to $\mu_a^{(s)}$, *c.f.* (3.39), ensures that

$$\frac{e^{-2i\pi[\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu_a^{(s)}) - \widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}((\beta_a^{(s)})^*)]} - 1}{1 - e^{-2i\pi\widehat{\xi}_{\Lambda^{(\alpha)}}^{(b)}(\mu_a^{(s)})}} = O(L^{-\infty}) \quad \text{so that} \quad \Re_F^{-1} = 1 + O(n_{\text{sg}} \cdot L^{-\infty}). \quad (6.17)$$

The contour integral can be decomposed as

$$\oint_{\mathcal{C}} \widehat{z}(s) \ln' [e^{-2i\pi\widehat{F}^{(b)}(s)} - 1] \cdot \frac{ds}{2i\pi} = \int_{\mathcal{C}^{(+)}} \widehat{F}^{(b)}(s) \ln' [e^{-2i\pi\widehat{F}^{(b)}(s)} - 1] \cdot ds + \Im_L \quad (6.18)$$

where the remainder takes the form

$$\Im_L = \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} (\widehat{u}_{\Gamma}^{(\epsilon)} - \widehat{u}_{\Lambda_b}^{(\epsilon)})(s) \cdot \ln' [e^{-2i\pi\widehat{F}^{(b)}(s)} - 1] \cdot \frac{ds}{2i\pi}. \quad (6.19)$$

The remainder term is a $O(L^{-1})$ as can be seen by invoking the $L^1(\mathcal{C}^{(\epsilon)})$ bounds (B.14) on $\widehat{u}_{\Gamma}^{(\epsilon)}$, $\widehat{u}_{\Lambda_b}^{(\epsilon)}$, the lower bound (3.114), the estimates (3.52), (B.7) and (3.100).

The first term in the *rhs* of (6.18) can be computed in closed form. Indeed, one has

$$\oint_{\mathcal{C}^{(+)}} \widehat{F}^{(b)}(s) \ln' [e^{-2i\pi\widehat{F}^{(b)}(s)} - 1] \cdot ds = -i\frac{\pi}{2} \left((\widehat{F}^{(b)}(\widehat{q}_L))^2 - (\widehat{F}^{(b)}(\widehat{q}_R))^2 \right) + \int_{\widehat{F}^{(b)}(\widehat{q}_R)}^{\widehat{F}^{(b)}(\widehat{q}_L)} \pi x \cot(\pi x) dx. \quad (6.20)$$

Then, the integral representation for the ratio of Barnes functions (A.2) allows one to conclude that

$$\begin{aligned} \exp \left\{ \oint_{\mathcal{C}} \widehat{z}(s) \ln' [e^{-2i\pi \widehat{F}^{(b)}(s)} - 1] \cdot \frac{ds}{2i\pi} \right\} \\ = \prod_{v \in \{L, R\}} \left\{ \frac{e^{i\frac{\pi}{2} \sigma_v (\widehat{F}^{(b)}(\widehat{q}_v))^2}}{(2\pi)^{\sigma_v \widehat{F}^{(b)}(\widehat{q}_v)}} \cdot G \left(\begin{matrix} 1 + \sigma_v \widehat{F}^{(b)}(\widehat{q}_v) \\ 1 - \sigma_v \widehat{F}^{(b)}(\widehat{q}_v) \end{matrix} \right) \right\} \cdot \left(1 + O\left(\frac{1}{L}\right)\right). \end{aligned} \quad (6.21)$$

Recalling the reflection identity (A.3) satisfied by the Barnes function and upon denoting $\widehat{\mathbf{f}}_{v, \text{tot}}^{(b)} = \kappa_v - \widehat{F}^{(b)}(\widehat{q}_v)$, one obtains the rewriting

$$\begin{aligned} \prod_{v \in \{L, R\}} \left\{ (e^{-2i\pi \widehat{F}^{(b)}(\widehat{q}_v)} - 1)^{\sigma_v \kappa_v} G \left(\begin{matrix} 1 + \sigma_v \widehat{F}^{(b)}(\widehat{q}_v) \\ 1 - \sigma_v \widehat{F}^{(b)}(\widehat{q}_v) \end{matrix} \right) \right\} \\ = \prod_{v \in \{L, R\}} \left\{ \frac{(2i\pi)^{\sigma_v \kappa_v} (-1)^{\frac{\kappa_v(\kappa_v-1)}{2}}}{e^{-i\pi \sigma_v \kappa_v \widehat{\mathbf{f}}_{v, \text{tot}}^{(b)}}} \cdot G \left(\begin{matrix} 1 - \sigma_v \widehat{\mathbf{f}}_{v, \text{tot}}^{(b)} \\ 1 + \sigma_v \widehat{\mathbf{f}}_{v, \text{tot}}^{(b)} \end{matrix} \right) \right\} \\ = \prod_{v \in \{L, R\}} \left\{ \frac{(2i\pi)^{\sigma_v \kappa_v} (-1)^{\frac{\kappa_v(\kappa_v-1)}{2}}}{e^{-i\pi \sigma_v \kappa_v \widehat{\mathbf{f}}_v^{(b)}}} \cdot G \left(\begin{matrix} 1 - \sigma_v \widehat{\mathbf{f}}_v^{(\varrho)} \\ 1 + \sigma_v \widehat{\mathbf{f}}_v^{(\varrho)} \end{matrix} \right) \right\} \cdot \left(1 + O\left(\frac{1}{L}\right)\right). \end{aligned} \quad (6.22)$$

The last line follows as a consequence of (3.100) which entails $|\widehat{\mathbf{f}}_{v, \text{tot}}^{(b)} - \widehat{\mathbf{f}}_v^{(\varrho)}| = O(L^{-1} + n_{\text{sg}} L^{-\infty})$ and, by construction, $|\widehat{\mathbf{f}}_v| \leq 1/2$ one has $|\widehat{\mathbf{f}}_v^{(\varrho)}| \leq 3/4$ what ensures that one is far from the poles and zeroes of the Barnes functions. By putting all the estimates together, the claim follows. \blacksquare

Proposition 6.4. *There exists r such that the conclusions of Lemma 3.8 hold and such that*

$$\det [\text{id} + U_{\alpha; \theta}^{(\gamma)} [F^{(\varrho)}]] \neq 0 \quad \text{for any } \varrho \in \partial \mathcal{D}_{0, r}. \quad (6.23)$$

Let $r_L = r + \delta r/L$ with δr as given by Proposition 3.2 and $\mathfrak{b} = \varrho + \delta \varrho/L \in \mathcal{D}_{0, r_L} \setminus \{\pm r_L\}$. Assume that the string centre separation hypothesis holds (3.37). Then one has the asymptotic behaviour

$$\widehat{C}^{(\gamma)}[\widehat{F}^{(b)}](\Lambda_{\mathfrak{b}}^{(\alpha)}; \Upsilon) = C^{(\gamma)}(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^{\mathfrak{C}}\} | \varrho) \cdot \left\{ 1 + O\left(\frac{\vartheta + 1}{L}\right) \right\} \quad (6.24)$$

where $c \geq 0$, $\gamma \in \{z, +\}$ and $C^{(\gamma)}$ are defined by (4.17)-(4.18).

Proof —

Let $r_L = r + \delta r/L$ with r such that the conclusions of Lemma 3.8 hold and with the subordinate δr given by Proposition 3.2.

The form taken by the pre-factors follows from the asymptotic expansion of $\widehat{\mathcal{P}}_{\Upsilon \setminus \Lambda_{\mathfrak{b}}^{(\alpha)}}^{(b)}$ (3.102) and from Corollary 5.2. The Fredholm determinant part deserves, however, more attention. Lemma 3.8 ensures the existence of a small open neighbourhood \mathcal{V}_F of the interval I_q such that the only zeroes of $s \mapsto 1 - e^{2i\pi \widehat{F}^{(b)}(s)}$ in \mathcal{V}_F are given by $Z \cup W$, c.f. (3.110). Furthermore, there exists $\epsilon > 0$ such that for each $w_a \in W$ there exists z_{w_a} so that one has $w_a \in \mathcal{D}_{z_{w_a}, \epsilon}$ and no other zero w_b or element of $Z \cup \{Z^{(s)} - i\zeta\}$ is contained there. Finally, the discs are uniformly away from the endpoints $\pm q$. These pieces of information allow one to relate the original Fredholm determinants to ones involving other operators whose thermodynamic limit is, however, easier to cope with.

By deforming contours in the Fredholm series representation for $\det_{\Gamma(\Lambda_b^{(\alpha)})} [\text{id} + \widehat{U}_{\alpha;\theta}^{(\gamma)}]$ from the small loop $\Gamma(\Lambda_b^{(\alpha)})$ around the roots $\Lambda_b^{(\alpha)}$ up to the boundary $\partial\mathcal{V}_F$ one picks up poles at $W \cup Z$. Here, one should note that although the factors $(V_{\Upsilon, \Lambda_b^{(\alpha)}})^{-1}(\tau + i\zeta)$ contain poles at the elements of $Z^{(s)} - i\zeta$, this does not generate poles of the integral kernel since these singularities are compensated by the poles of $e^{2i\pi\widehat{F}^{(b)}(\tau)}$. This contour deformation yields

$$\det_{\Gamma(\Lambda_b^{(\alpha)})} [\text{id} + \widehat{U}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]] = \det [\text{id} + \widehat{W}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]] \quad (6.25)$$

where $\widehat{W}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]$ is the operator on $L^2(\partial\mathcal{V}_F \cup W \cup Z)$:

$$(\widehat{W}_{\alpha;\theta}^{(\gamma)} \cdot f)(\omega) = \int_{\partial\mathcal{V}_F} \widehat{U}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \tau) f(\tau) \cdot d\tau - 2i\pi \sum_{\mu \in W \cup Z} f(\mu) \text{Res}_{\tau=\mu} \left(\widehat{U}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \tau) d\tau \right). \quad (6.26)$$

One can then rewrite the functions $V_{\Upsilon, \Lambda_b^{(\alpha)}}$ appearing in the integral kernel $\widehat{U}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]$ by using Corollary 5.2 and the fact that either τ belongs to the exterior of \mathcal{C} or that it belongs to the interior but coincides with a zero of $1 - e^{2i\pi\widehat{F}^{(b)}(s)}$. This means that one can replace the integral kernel $\widehat{U}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]$ by

$$\widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \tau) = \frac{e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\tau)}}{e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\tau+i\zeta)}} \prod_{\mu \in \Upsilon_{\text{tot}}^{(z)} \cup \Upsilon \setminus \Upsilon^{(h)}} \left\{ \frac{\sinh(\tau - \mu)}{\sinh(\tau - \mu + i\zeta)} \right\} \cdot \frac{\mathcal{K}_{\alpha;\theta}^{(\gamma)}(\omega, \tau)}{1 - e^{2i\pi\widehat{F}^{(b)}(\tau)}}. \quad (6.27)$$

The resulting kernel is a meromorphic function of τ outside and inside of \mathcal{C} . The poles inside of \mathcal{C} are located[†] at $\Upsilon^{(h)} \cup W \cup Z$. Hence, once that the kernels are replaced, one can reabsorb the residues at W by the contour integrals along $\bigcup_{a=1}^{\ell_F} \partial\mathcal{D}_{z_w, \epsilon}$. The above stated properties of these discs ensure that, one does not produce new residues when doing so: all the other poles of the integrand are located outside of these discs. Hence, one obtains

$$\det_{\Gamma(\Lambda_b^{(\alpha)})} [\text{id} + \widehat{U}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]] = \det_{\mathfrak{g} \cup \llbracket 1; n_{\text{sg}} \rrbracket} [\text{id} + \widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]]. \quad (6.28)$$

Here $\mathfrak{g} = \partial\mathcal{V}_F \setminus \bigcup_{a=1}^{\ell_F} \mathcal{D}_{z_w, \epsilon}$ and $\widehat{V}_{\alpha;\theta}^{(\gamma)}$ is the integral operator on $L^2(\mathfrak{g} \cup \llbracket 1; n_{\text{sg}} \rrbracket)$:

$$(\widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}] \cdot f)(\omega) = \int_{\mathfrak{g}} \widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \tau) f(\tau) \cdot d\tau + \sum_{a=1}^{n_{\text{sg}}} \delta \widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \mathfrak{z}_a^{(s)}) f_a. \quad (6.29)$$

where

$$\delta \widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \tau) = \frac{e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\tau)}}{e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\tau+i\zeta)}} \prod_{\mu \in \Upsilon_{\text{tot}}^{(z)} \cup \Upsilon \setminus \Upsilon^{(h)}} \left\{ \frac{\sinh(\tau - \mu)}{\sinh(\tau - \mu + i\zeta)} \right\} \cdot \frac{\mathcal{K}_{\alpha;\theta}^{(\gamma)}(\omega, \tau)}{(\widehat{F}^{(b)})'(\tau)}. \quad (6.30)$$

It is also useful to introduce a second integral operator on $L^2(\mathfrak{g} \cup \llbracket 1; n_{\text{sg}} \rrbracket)$:

$$(\mathbf{V}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}] \cdot f)(\omega) = \int_{\mathfrak{g}} U_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}](\omega, \tau) f(\tau) \cdot d\tau \quad (6.31)$$

[†] Here, the poles/zeros issuing from the product over Υ cancel with the contributions issuing from $\Upsilon^{(z)}$.

whose integral kernel has been defined in (4.16). The operators $\widehat{V}_{\alpha;\theta}^{(\gamma)}$ and $V_{\alpha;\theta}^{(\gamma)}$ have C^1 kernel and act on functions supported on compacts. They are therefore trace class in virtue of the results of [11]. This ensures that they are Hilbert-Schmidt as well. Also, they admit the matrix representation subordinate to the splitting $\mathfrak{g} \cup \llbracket 1; n_{\text{sg}} \rrbracket$:

$$\widehat{V}_{\alpha;\theta}^{(\gamma)} = \begin{pmatrix} \widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \tau) & \widehat{\delta V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\omega, \mathfrak{z}_b^{(s)}) \\ \widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\mathfrak{z}_a^{(s)}, \tau) & \widehat{\delta V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\mathfrak{z}_a^{(s)}, \mathfrak{z}_b^{(s)}) \end{pmatrix} \quad (6.32)$$

and

$$V_{\alpha;\theta}^{(\gamma)} = \begin{pmatrix} U_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}](\omega, \tau) & 0 \\ 0 & 0 \end{pmatrix}. \quad (6.33)$$

Direct bounds on $\mathfrak{g} \cup Z$ yield

$$\left| e^{2i\pi[C_{I_q}[F^{(\varrho)}](\tau) - 2i\pi C_{I_q}[F^{(\varrho)}](\tau + i\zeta)} \cdot G(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^{\mathbb{Z}}\} \mid \tau) \right| \leq e^c \quad (6.34)$$

for some constant $c \geq 0$. This constant vanishes if the function has modulus lower than one. Then, by using the estimates (3.114), the results of Corollary 5.2 and the fact guaranteed by Lemma 3.8 that for any root w_a , one has $d(z_{w_a}, \{Z^{(s)} - i\zeta\} \cup \{\pm q\}) > 2\epsilon$, one can check that

$$\|\widehat{V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}] - U_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]\|_{L^\infty(\{\mathfrak{g} \cup Z\} \times \mathfrak{g})} = O\left(\frac{\mathfrak{U} + 1}{L}\right). \quad (6.35)$$

In the estimation of the large- L behaviour of the kernel $\widehat{\delta V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\lambda, \mathfrak{z}_b^{(s)})$ one should pay an extra attention to the string containing the root $\mu_a^{(s)}$ which approaches $\mathfrak{z}_a^{(s)}$ exponentially fast. The contribution of this string cancel out with the associated singular factor appearing in the product over \mathfrak{T} while the string deviations appearing in the other only produce exponentially small corrections owing to the string spacing hypothesis (3.37). This hypothesis also ensures that

$$(\widehat{F}^{(b)})'(\mathfrak{z}_a^{(s)}) = \frac{-1}{2\pi} \frac{\sin[2\mathfrak{I}(\beta_a^{(s)})]}{\sinh(\mathfrak{z}_a^{(s)} - \beta_a^{(s)}) \sinh(\mathfrak{z}_a^{(s)} - (\beta_a^{(s)})^*)} (1 + O(n_{\text{tot}} L^{-\infty})) \quad (6.36)$$

so that, for any $\lambda \in \mathfrak{g} \cup Z$ with $Z = \{\mathfrak{z}_a^{(s)}\}_1^{n_{\text{sg}}}$,

$$\begin{aligned} \widehat{\delta V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}](\lambda, \mathfrak{z}_a^{(s)}) &= -2\pi \frac{e^{2i\pi C_{I_q^\uparrow}[F^{(\varrho)}](\mathfrak{z}_a^{(s)})}}{e^{2i\pi C_{I_q}[F^{(\varrho)}](\mathfrak{z}_a^{(s)} + i\zeta)}} G(\Upsilon_{\text{off}}^{(h)}; \mathfrak{C}; \{\ell_v^{\mathbb{Z}}\} \mid \mathfrak{z}_a^{(s)}) \cdot \mathcal{K}_{\alpha;\theta}^{(\gamma)}(\omega, \mathfrak{z}_a^{(s)}) \\ &\quad \times \frac{\sinh(\mathfrak{z}_a^{(s)} - \beta_a^{(s)}) \sinh(\mathfrak{z}_a^{(s)} - (\beta_a^{(s)})^*)}{\sin[2\mathfrak{I}(\beta_a^{(s)})]} (1 + O\left(\frac{\mathfrak{U} + 1}{L}\right)). \end{aligned} \quad (6.37)$$

Above, I_q^\uparrow is a small deformation of I_q which avoids $\mathfrak{z}_a^{(s)}$ from above. Therefore, owing to the estimates (3.112),

$$\|\widehat{\delta V}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(b)}]\|_{L^\infty(\{\mathfrak{g} \cup Z\} \times Z)} = O(L^{-\infty}). \quad (6.38)$$

Prior to obtaining the bounds, it remains to establish the statement about the non-vanishing of the determinant. The integral kernel $U_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]$ is an analytic function of ρ on the annulus $r_1 > |\rho| > r_2$

with r_1, r_2 as given in Lemma 3.8. Hence, so is $\det_{\mathfrak{g}}[\text{id} + \mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]]$. Thus, one can always pick some $r \in [r_2; r_1]$ such that $|\det_{\mathfrak{g}}[\text{id} + \mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]]| > c > 0$ on $\partial\mathcal{D}_{0,r}$. This choice of r , and the subordinate value of δr , will be assumed in the following.

The Lipschitz bound for 2-determinants [17] of Hilbert-Schmidt operators \mathbf{A}, \mathbf{B}

$$\left| \det_2[\text{id} + \mathbf{A}] - \det_2[\text{id} + \mathbf{B}] \right| \leq \|\mathbf{A} - \mathbf{B}\|_2 \cdot e^{C(\|\mathbf{B}\|_2 + \|\mathbf{A}\|_2)} \quad (6.39)$$

yields the below estimate for two trace class operators \mathbf{A}, \mathbf{B}

$$\left| \det[\text{id} + \mathbf{A}] - \det[\text{id} + \mathbf{B}] \right| \leq \left\{ \|\mathbf{A} - \mathbf{B}\|_2 + |\text{tr}(\mathbf{A} - \mathbf{B})| \right\} \cdot e^{C(\|\mathbf{B}\|_2 + \|\mathbf{A}\|_2 + |\text{tr}(\mathbf{A})| + |\text{tr}(\mathbf{B})| + 1)}. \quad (6.40)$$

One has

$$\left| \text{tr}[\widehat{\mathbf{v}}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(\mathfrak{b})}] - \mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]] \right| + \left\| \widehat{\mathbf{v}}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(\mathfrak{b})}] - \mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}] \right\|_2 = O\left(\frac{\mathfrak{U} + 1}{L}\right) \quad (6.41)$$

either directly in virtue of the bounds (6.35)-(6.38) or due to the fact that when calculating the Hilbert-Schmidt norm, the last line of the matrix representation (6.32) for $\widehat{\mathbf{v}}_{\alpha;\theta}^{(\gamma)}$, which contains *a priori* order 1 terms, always appears in pair with the last column in (6.32) which is exponentially small. Also, I used the upper bound $\ell_F \leq C n_{\text{tot}}^{(\text{msv})}$ on the number of zeroes of $1 - e^{2i\pi F^{(\varrho)}}$ inside of \mathcal{V}_F . Similarly, one can check that

$$\left| \text{tr}[\mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]] \right| + \left\| \mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}] \right\|_2 = O(1). \quad (6.42)$$

Therefore, by using the lower bound on the Ferholm determinant, one gets that

$$\det_{\mathfrak{g} \cup \llbracket 1; n_{\text{sg}} \rrbracket} [\text{id} + \widehat{\mathbf{v}}_{\alpha;\theta}^{(\gamma)}[\widehat{F}^{(\mathfrak{b})}]] = \det_{\mathfrak{g}} [\text{id} + \mathbf{v}_{\alpha;\theta}^{(\gamma)}[F^{(\varrho)}]] \left\{ 1 + O\left(\frac{\mathfrak{U} + 1}{L}\right) \right\}. \quad (6.43)$$

It remains to gather the bounds. ■

7 Analysis of $\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon)$

This section discusses the large- L expansion of $\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon)$. The most delicate part consists in extracting the large- L behaviour out of the "norm"-determinants, this while keeping a satisfactory control on the remainder.

Proposition 7.1. *The below large- L asymptotic expansion holds*

$$\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon) = \frac{1}{\prod_{r=2}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \{L p'_r(c_a^{(r)})\}} \cdot \left(1 + O\left(\frac{\mathfrak{U} + 1}{L}\right) \right) \quad (7.1)$$

Proof — The claim follows by straightforward handlings of the results obtained in Lemma 7.4. ■

7.1 An auxiliary determinant identity

One of the key results which allows one to carry out the large- L analysis of $\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon)$ is an auxiliary determinant identity which allows one to recast the "norm"-determinants in a form that allows one to extract their large- L asymptotics in the presence of string solutions, this while providing a control on the corrections.

Lemma 7.2. *Let*

$$\Delta(P_a; X_{ab}) = \det_N \left[\delta_{ab} \left(P_a - \sum_{k=1}^N X_{ak} \right) + X_{ab} \right] \quad (7.2)$$

be a determinant defined in terms of an $N \times N$ symmetric matrix X_{ab} having a vanishing diagonal $X_{aa} = 0$. Then it holds

$$\Delta(P_a; X_{ab}) = \det_N \left[\sum_{a=\max(k,\ell)}^N P_a - \sum_{s=1}^{\min(k,\ell)-1} \sum_{t=\max(k,\ell)}^N X_{st} \right] \quad (7.3)$$

Proof —

This rewriting of the original determinant can be obtained by doing the chain of linear combinations on the lines L_a and columns C_a : $L_a \hookrightarrow \sum_{k=a}^N L_k$ and $C_a \hookrightarrow \sum_{k=a}^N C_k$ with $a = 1, 2, \dots, N-1$. The claim is proven by induction, where the induction hypothesis is $\Delta(P_a; X_{ab}) = \det_N [M^{(j)}]$ for any j with

$$M^{(j)} = \begin{pmatrix} \sum_{a=\max(k,\ell)}^N P_a & - \sum_{s=1}^{\min(k,\ell)-1} \sum_{t=\max(k,\ell)}^N X_{st} & P_\ell - \sum_{s=1}^{k-1} X_{ks} \\ P_k - \sum_{s=1}^{\ell-1} X_{ks} & \delta_{k\ell} \left(P_k - \sum_{s=1}^N X_{ks} \right) + X_{k\ell} \end{pmatrix}. \quad (7.4)$$

Above, the matrix is written in a block decomposition subordinate to $k, \ell \leq j$ for the first diagonal and $k, \ell \geq j+1$ for the second diagonal. The details are left to the reader.

Proposition 7.3. *Let*

$$\Delta(P_a; X_{ab}) = \det_{n_\Upsilon} \left[\delta_{ab} \left(P_a - \sum_{k=1}^N X_{ak} \right) + X_{ab} \right] \quad (7.5)$$

be a determinant defined in terms of an $n_\Upsilon \times n_\Upsilon$ symmetric matrix X_{ab} having a vanishing diagonal $X_{aa} = 0$ and let $n_\Upsilon = \sum_{r=2}^{p_{\max}} r n_r^{(z)}$. Let

$$v_{p,a,k} = \sum_{r=2}^{p-1} r n_r^{(z)} + (a-1)p + k \quad (7.6)$$

be a linear index labelling a block matrix decomposition into blocks labelled by the length r , ranging over $2, \dots, p_{\max}$, sub-blocks of fixed length r but having indices $a = 1, \dots, n_r^{(z)}$ and, finally, $r \times r$ sub-blocks labelled by $k = 1, \dots, r$. Let

$$\mathfrak{P}_k^{(p,a)} = P_{v_{p,a,k}} - \sum_{\substack{t \neq v_{p,a,k} \\ k=1, \dots, p}}^{n_\Upsilon} X_{t, v_{p,a,k}}, \quad \mathfrak{X}_{k,\ell}^{(p,a)} = X_{v_{p,a,k}, v_{p,a,\ell}} \quad \text{and} \quad \mathfrak{X}_{k,\ell}^{(p,a),(r,b)} = X_{v_{p,a,k}, v_{r,b,\ell}}. \quad (7.7)$$

Then one has the block matrix decomposition

$$\Delta(P_a; X_{ab}) = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (7.8)$$

The block matrix A has its entries labelled by the 2-ple (p, a)

$$A_{(p,a),(r,b)} = \overline{\mathfrak{P}}_1^{(p,a)} \cdot \delta_{(p,a),(r,b)} - (1 - \delta_{(p,a),(r,b)}) \cdot \overline{\mathfrak{X}}_{1,1}^{(p,a),(r,b)} \quad \text{with} \quad \begin{cases} \overline{\mathfrak{P}}_k^{(p,a)} = \sum_{s=k}^p \mathfrak{P}_s^{(p,a)} \\ \overline{\mathfrak{X}}_{k,\ell}^{(p,a),(r,b)} = \sum_{s=k}^p \sum_{t=\ell}^r \mathfrak{X}_{s,t}^{(p,a),(r,b)} \end{cases}. \quad (7.9)$$

The Kronecker symbol $\delta_{(p,a),(r,b)}$ appearing above is such that $\delta_{(p,a),(r,b)} = 1$ if the 2-uples coincide and zero otherwise. The off-diagonal blocks B and C read

$$B_{(p,a),(r,b,k)} = -\overline{\mathfrak{X}}_{1,k}^{(p,a),(r,b)}, \quad C_{(p,a,k),(r,b)} = -\overline{\mathfrak{X}}_{k,1}^{(p,a),(r,b)} \quad (7.10)$$

where (p, a) and (r, b) run through $\{(w, t) : w = 2, \dots, p_{\max}, t = 1, \dots, n_w^{(z)}\}$ and k in (p, a, k) , resp. (r, b, k) , $k = 2, \dots, p$, resp. $k = 2, \dots, r$. Finally, the D block takes the form

$$D_{(p,a,k),(r,b,\ell)} = \left[\overline{\mathfrak{P}}_1^{(p,a)} - \sum_{s=1}^{\min(\ell,k)-1} \sum_{t=\max(\ell,k)}^p \mathfrak{X}_{s,t}^{(p,a)} \right] \cdot \delta_{(p,a),(r,b)} - (1 - \delta_{(p,a),(r,b)}) \cdot \overline{\mathfrak{X}}_{k,\ell}^{(p,a),(r,b)}. \quad (7.11)$$

In particular, if the matrix A is invertible, then one has

$$\Delta(P_a; X_{ab}) = \det A \cdot \prod_{r=2}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \prod_{k=1}^{r-1} \left\{ -\mathfrak{X}_{k,k+1}^{(p,a)} \right\} \cdot \left\{ 1 + O \left(n_Y \frac{\max_{(r,b,\ell) \neq (r,b,k \pm 1)} X_{k,\ell}^{(p,a),(r,b)}}{\min \mathfrak{X}_{k,k+1}^{(p,a)}} \cdot \max(A^{-1})_{(p,a),(r,b)} \right) \right\}. \quad (7.12)$$

A similar result, but without the explicit control of the remainder was obtained in [22]. However, the technique developed in [22] does not allow one for any estimate on the remainder. This control is important since the matrix $\delta_{ab}(P_a - \sum_{k=1}^N X_{ak}) + X_{ab}$ of interest to the analysis will have, in principle, various entries that will diverge. Some at exponential speed and some at algebraic. The control in (7.12) allows one to neglect the contribution of the entries diverging only algebraically in L .

Proof —

The block matrix decomposition (7.8) is obtained, first, by doing linear combinations of lines and columns $L_{v_{a,p,k}} \hookrightarrow \sum_{s=k}^p L_{v_{a,p,s}}$ and $C_{v_{a,p,k}} \hookrightarrow \sum_{s=k}^p C_{v_{a,p,s}}$ for $k = 1, \dots, p-1$ and (p, a) fixed and this for each (a, p) index. The form of the diagonal (a, p) block is taken care of by Lemma 7.2 whereas the form of the off-diagonal blocs is given by a simple sum. It then solely remains to exchange appropriate lines and columns of the determinant so that the $((p, a, 1), (r, b, 1))$ entry of each diagonal bloc moves to the $((p, a), (r, b))$ entry of the resulting matrix, hence giving rise to A . The estimate on the remainder follows from (6.40). ■

7.2 The large- L analysis

In order to recast $\widehat{\mathcal{A}}_{\text{sing}}(\Upsilon)$ in a form allowing one to take the thermodynamic limit readily, the main point is to rewrite $\det[\Xi_\Upsilon]$ in a different form. Such manipulations, eventually, lead to

Lemma 7.4. Let $\Upsilon_{\text{red}} = \Upsilon \setminus \Upsilon^{(z)}$. It holds for L large enough:

$$\det[\Xi_{\Upsilon}] = \det[\Xi_{\Upsilon_{\text{red}}}] \cdot \frac{\prod_{r=2}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \left\{ \frac{L}{2\pi} p'_r(c_a^{(r)}) \right\}}{\prod_{\mu \in \Upsilon^{(z)}} \left\{ L \widehat{\xi}'_{\Upsilon}(\mu) \right\}} \cdot \prod_{r=2}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \prod_{k=1}^{r-1} \left\{ -K(\delta_{a,k}^{(r)} - \delta_{a,k+1}^{(r)} + i\zeta) \right\} \cdot \left\{ 1 + O\left(\frac{1}{L} n_{\text{tot}}^{(z)}\right) \right\} \quad (7.13)$$

where

$$[\Xi_{\Upsilon_{\text{red}}}]_{ab} = \delta_{ab} + \frac{K(v_a - v_b)}{L \widehat{\xi}'_{\Upsilon}(v_b)}. \quad (7.14)$$

Proof—

Consider the parametrisation

$$\Upsilon \setminus \Upsilon^{(z)} \equiv \Upsilon_{\text{red}} = \{v_a\}_1^{|\Upsilon_{\text{red}}|}, \quad \Upsilon^{(z)} = \{z_a\}_1^{|\Upsilon^{(z)}|}. \quad (7.15)$$

Clearly, one has $\det[\Xi_{\Upsilon}] = \det[\widetilde{\Xi}]$, where $\widetilde{\Xi}$ is an $|\Upsilon| \times |\Upsilon|$ matrix with which takes the form

$$\widetilde{\Xi} = \begin{pmatrix} [\Xi_{\Upsilon_{\text{red}}}]_{ab} & \widetilde{K}(v_a, z_b) \\ \widetilde{K}(z_a, v_b) & \delta_{ab} + \widetilde{K}(z_a, z_b) \end{pmatrix} \quad \text{with} \quad \widetilde{K}(\lambda, \mu) = \frac{K(\lambda - \mu)}{L \widehat{\xi}'_{\Upsilon}(\mu)}. \quad (7.16)$$

It follows from Lemma 7.6 that the matrix $\Xi_{\Upsilon_{\text{red}}}$ is invertible. Its inverse can be represented in the form

$$[\Xi_{\Upsilon_{\text{red}}}^{-1}]_{ab} = \delta_{ab} - \frac{\widehat{R}(v_a, v_b)}{L \widehat{\xi}'_{\Upsilon}(v_b)}. \quad (7.17)$$

The so-called discrete resolvent \widehat{R} is a function on the discrete set $\Upsilon_{\text{red}} \times \Upsilon_{\text{red}}$. One can extend the definition of this function almost everywhere to \mathbb{C}^2 , first for $\tau \in \Upsilon_{\text{red}}$ and z a generic complex number by the formula:

$$\widehat{R}(\tau, z) = K(\tau - z) - \sum_{\alpha \in \Upsilon_{\text{red}}} \frac{\widehat{R}(\tau, \alpha) K(\alpha - z)}{L \widehat{\xi}'_{\Upsilon}(\alpha)}, \quad (7.18)$$

and then, for z, z' generic, by

$$\widehat{R}(z, z') = K(z - z') - \sum_{\alpha \in \Upsilon_{\text{red}}} \frac{K(z - \alpha) \widehat{R}(\alpha, z')}{L \widehat{\xi}'_{\Upsilon}(\alpha)}. \quad (7.19)$$

$\widehat{R}(z, z')$ constitutes the discrete version of the continuous resolvent operator R , see Lemma 7.7 for a precise statement.

Owing to the factorisation for block determinants

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det[A] \cdot \det[D - CA^{-1}B] \quad (7.20)$$

valid whenever the block matrix A is invertible, one gets the determinant factorisation

$$\det[\Xi_{\Upsilon}] = \det[\Xi_{\Upsilon_{\text{red}}}] \frac{\det_{n_{\Upsilon}}[\widehat{Y}]}{\prod_{\mu \in \Upsilon^{(z)}} \left\{ L \widehat{\xi}'_{\Upsilon}(\mu) \right\}} \quad \text{with} \quad n_{\Upsilon} = |\Upsilon^{(z)}| \quad (7.21)$$

where the matrix \widehat{Y}_{ab} has entries $\widehat{Y}_{ab} = L\widehat{\xi}'_{\Upsilon}(z_a)\delta_{ab} + \widehat{R}(z_a, z_b)$. The function

$$\widehat{\rho}(\omega) = K(\omega \mid \zeta/2) - \sum_{\alpha \in \Upsilon_{\text{red}}} \widehat{R}(\omega, \alpha) K(\alpha \mid \zeta/2) \quad (7.22)$$

allows one to recast $\widehat{\xi}'_{\Upsilon}$ in the form

$$L\widehat{\xi}'_{\Upsilon}(\omega) = L\widehat{\rho}(\omega) - \sum_{\alpha \in \Upsilon^{(z)}} \widehat{R}(\omega, \alpha). \quad (7.23)$$

The above representation for the counting function thus ensures that the matrix \widehat{Y} takes the generic form

$$\widehat{Y}_{ab} = \delta_{ab} \left(P_a + \sum_{k=1}^{n_{\Upsilon}} X_{ak} \right) - X_{ab} \quad \text{for } a, b = 1, \dots, n_{\Upsilon} \quad (7.24)$$

for an appropriate choice of the symmetric matrix X_{ab} and the P_a 's. The matrix \widehat{Y} has thus precisely the form taken care of by Proposition 7.3. In the notations of that proposition, one has

$$\overline{\mathfrak{P}}_1^{(p,a)} = L(\widehat{\xi}'_{\Upsilon;a})'(c_a^{(r)}) \quad \text{and} \quad \overline{\mathfrak{X}}_{1,1}^{(p,a)(r,b)} = \widehat{\mathcal{R}}_{ab}^{(pr)}(c_a^{(p)}, c_b^{(r)}) \quad (7.25)$$

where the diagonal entries are expressed in terms of

$$\left(\widehat{\xi}'_{\Upsilon;a}^{(r)} \right)'(\omega) = \sum_{k=1}^r \widehat{\xi}'_{\Upsilon \setminus \mathcal{S}_a^{(r)}}(\omega + i\frac{\zeta}{2}(r+1-2k) + \delta_{a,k}^{(r)}) \quad (7.26)$$

where

$$\mathcal{S}_a^{(r)} = \left\{ c_a^{(r)} + i\frac{\zeta}{2}(r+1-2k) + \delta_{a,k}^{(r)} \right\}_{k=1}^r \quad (7.27)$$

and

$$\widehat{\xi}'_{\Upsilon \setminus \mathcal{S}_a^{(r)}}(\omega) = K(\omega \mid \frac{\zeta}{2}) - \frac{1}{L} \sum_{\alpha \in \Upsilon \setminus \mathcal{S}_a^{(r)}} K(\omega - \alpha). \quad (7.28)$$

The off-diagonal entries are expressed in terms of the function

$$\widehat{R}_{ab}^{(rq)}(\lambda, \mu) = \sum_{k=1}^r \sum_{\ell=1}^q \widehat{R}\left(\lambda + i\frac{\zeta}{2}(r+1-2k) + \delta_{a,k}^{(r)}, \mu + i\frac{\zeta}{2}(q+1-2\ell) + \delta_{b,\ell}^{(q)}\right). \quad (7.29)$$

It follows from Lemma 7.5 and Lemma 7.7 that the matrix $\overline{\mathfrak{P}}_1^{(p,a)} \delta_{(p,a),(r,b)} - \overline{\mathfrak{X}}_{1,1}^{(p,a)(r,b)} (1 - \delta_{(p,a),(r,b)})$ is invertible. Hence, after pulling out from the determinant the diagonal terms, one gets

$$\begin{aligned} \det[\widehat{Y}] &= \prod_{r=2}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \prod_{k=1}^{r-1} \left\{ -K(\delta_{a,k}^{(r)} - \delta_{a,k+1}^{(r)} + i\zeta) \right\} \\ &\quad \times \prod_{r=2}^{p_{\max}} \prod_{a=1}^{n_r^{(z)}} \left\{ L(\widehat{\xi}'_{\Upsilon;a})'(c_a^{(r)}) \right\} \cdot \det[\widehat{Y}_{\text{red}}] \cdot \left(1 + \mathcal{O}(n_{\text{tot}}^{(z)} \cdot L^{-\infty}) \right). \end{aligned} \quad (7.30)$$

The reduced matrix takes the form

$$[\widehat{Y}_{\text{red}}]_{(a,r),(b,q)} = \delta_{(a,r),(b,q)} + \frac{\widehat{\mathcal{R}}_{ab}^{(rq)}(c_a^{(r)}, c_b^{(q)})}{L(\widehat{\xi}_{\Upsilon;a}^{(r)})'(c_a^{(r)})}. \quad (7.31)$$

In order to proceed further and obtain the leading asymptotics of the determinant, one needs to invoke the simplified expression for the discrete resolvent obtained in Lemma 7.7 and the one for the re-summed counting functions obtained in Lemma 7.5. Then formula (6.40) yields $\det[\widehat{Y}_{\text{red}}] = 1 + (n_{\text{tot}}^{(z)} \cdot L^{-1})$. ■

Lemma 7.5. *Assume that the Bethe roots satisfy to the string centre spacing hypotheses (3.38) and (3.37). Then, it holds that*

$$(\widehat{\xi}_{\Upsilon;a}^{(r)})'(c_a^{(r)}) = \frac{1}{2\pi} \cdot p_r'(c_a^{(r)}) + O\left(\frac{\Im + 1}{L}\right) \quad (7.32)$$

where p_r is the dressed momentum of an r -string introduced in (2.21). The remainder in (7.32) is uniform in respect to the various parameters at play.

Proof—

Let

$$\mathfrak{s}_{a,k}^{(r)} = c_a^{(r)} + i\frac{\zeta}{2}(r+1-2k) + \delta_{a,k}^{(r)} \quad \text{and} \quad \mathfrak{c}_{a,k}^{(r)} = c_a^{(r)} + i\frac{\zeta}{2}(r+1-2k) + \delta_{a, \lfloor \frac{r+1}{2} \rfloor}^{(r)} \quad (7.33)$$

denote components of the various strings depending on whether one kept -or not- the exponentially small string deviations.

It follows from the Taylor-integral expansion and from hypothesis (3.37) that for any $(a, r) \neq (b, s)$,

$$|K(\mathfrak{s}_{a,k}^{(r)} - \mathfrak{s}_{b,\ell}^{(s)}) - K(\mathfrak{c}_{a,k}^{(r)} - \mathfrak{c}_{b,\ell}^{(s)})| \leq |\delta_{a,k}^{(r)} - \delta_{b,\ell}^{(s)}| \sup_{t \in [0;1]} |K'(\mathfrak{c}_{a,k}^{(r)} - \mathfrak{c}_{b,\ell}^{(s)} + t[\delta_{a,k}^{(r)} - \delta_{b,\ell}^{(s)}])| \leq C' |\delta_{a,k}^{(r)} - \delta_{b,\ell}^{(s)}| L^{2\kappa}. \quad (7.34)$$

Similarly, the hypothesis on the lower bound (3.36) on roots spacings ensures that, for any $\alpha \in \Upsilon^{(\text{in})} \cup \Upsilon^{(p)}$, one has

$$|K(\mathfrak{s}_{a,k}^{(r)} - \alpha) - K(\mathfrak{c}_{a,k}^{(r)} - \alpha)| \leq C \cdot \begin{cases} |\delta_{a,k}^{(r)} - \delta_{a, \lfloor \frac{r+1}{2} \rfloor}^{(r)}|, & k \neq \frac{r+1}{2}, \frac{r-3}{2} \\ |\Im(\delta_{a, \frac{r-1}{2}}^{(r)})|^{1-\nu}, & k = \frac{r+1}{2} \end{cases} \quad (7.35)$$

where ν is as given in (3.36) and I made use of (3.39) in the intermediate bounds. Finally, hypothesis (3.38) one also has that

$$|K(\mathfrak{s}_{a,k}^{(r)} | \zeta/2) - K(\mathfrak{c}_{a,k}^{(r)} | \zeta/2)| \leq C' \cdot \begin{cases} |\delta_{a,k}^{(r)} - \delta_{a, \lfloor \frac{r+1}{2} \rfloor}^{(r)}| L^{2\kappa} & k \neq \frac{r}{2}, \frac{r}{2} + 1 \\ |\delta_{a, \frac{r}{2}+1}^{(r)} - \delta_{a, \frac{r}{2}}^{(r)}|^{1/2} & k = \frac{r}{2} + 1 \end{cases}. \quad (7.36)$$

This allows one to infer that

$$\widehat{\xi}'_{\Upsilon \setminus \mathcal{S}_a^{(r)}}(\mathfrak{s}_{a,k}^{(r)}) - \widehat{\xi}'_{\Upsilon \setminus \mathcal{S}_a^{(r)}}(\mathfrak{c}_{a,k}^{(r)}) = O(n_{\text{tot}} L^{-\infty}). \quad (7.37)$$

Thus using the summation identity (2.4) one gets that

$$(\widehat{\xi}_{\Upsilon;a}^{(r)})'(c_a^{(r)}) = \chi_a^{(r)}(\mathfrak{c}_{a, \lfloor \frac{r+1}{2} \rfloor}^{(r)}) + O(n_{\text{tot}}^2 L^{-\infty}) = \chi_a^{(r)}(c_a^{(r)}) + O(L^{-\infty}) \quad (7.38)$$

where

$$\chi_a^{(r)}(\omega) = K\left(\omega \mid r\frac{\zeta}{2}\right) - \frac{1}{L} \sum_{\substack{\alpha \in \Upsilon^{(\text{in})} \\ \cup \Upsilon^{(p)} \setminus \Upsilon^{(h)}}} K_{r,1}(\omega - \alpha) - \frac{1}{L} \sum_{s=2}^{p_{\max}} \sum_{\substack{b=1 \\ (s,b) \neq (r,a)}}^{n_s^{(z)}} K_{r,s}(\omega - c_b^{(s)}) \quad (7.39)$$

For real ω it holds

$$-\frac{1}{L} \sum_{\alpha \in \Upsilon^{(\text{in})}} K_{r,1}(\omega - \alpha) = - \int_{-q}^q K_{r,1}(\omega - s) p'(s) \cdot \frac{ds}{2\pi} + \Re_r(\omega) \quad (7.40)$$

and

$$\begin{aligned} \Re_r(\omega) = & - \left\{ \int_q^{\widehat{q}_R} + \int_{\widehat{q}_L}^{-q} \right\} K_{r,1}(\omega - s) p'(s) \cdot \frac{ds}{2\pi} + \int_{\widehat{q}_L}^{\widehat{q}_R} K_{r,1}(\omega - s) \left[\frac{p'(s)}{2\pi} - \widehat{\xi}'_{\Upsilon_{\text{reg}}}(s) \right] \cdot ds \\ & - \sum_{\epsilon = \pm} \epsilon \int_{\mathcal{C}'(\epsilon)} K_{r,1}(\omega - s) \left\{ (\widehat{u}_{\Upsilon}^{(\epsilon)})'(s) - \widehat{\xi}'_{\Upsilon_{\text{sing}}}(s) \cdot \delta_{\epsilon,+} \right\} \cdot \frac{ds}{2i\pi L} - \frac{1}{L} \sum_{\alpha \in \Upsilon} K_{r,1}(\omega - \alpha). \end{aligned} \quad (7.41)$$

Owing to the bounds (3.6), (B.7), the estimates (3.39), the asymptotic expansion of the counting function (3.49) given in Proposition 3.4, and the bounds (B.13), (B.14) one has that $\Re_r(\omega) = O(L^{-1})$ on $\mathcal{S}_\delta(\mathbb{R})$. Thence,

$$\left(\widehat{\xi}_{\Upsilon,a}^{(r)} \right)'(c_a^{(r)}) = K\left(c_a^{(r)} \mid r\frac{\zeta}{2}\right) - \int_{-q}^q K_{r,1}(c_a^{(r)} - s) \cdot p'(s) \cdot ds + \mathcal{R}_a^{(r)}(c_a^{(r)}) \quad (7.42)$$

with

$$\mathcal{R}_a^{(r)}(\omega) = \Re_r(\omega) - \frac{1}{L} \sum_{\substack{\alpha \in \Upsilon^{(p)} \\ \setminus \Upsilon^{(h)}}} K_{r,1}(\omega - \alpha) - \frac{1}{L} \sum_{s=2}^{p_{\max}} \sum_{\substack{b=1 \\ (s,b) \neq (r,a)}}^{n_s^{(s)}} K_{r,s}(\omega - c_b^{(s)}) = O\left(\frac{1}{L}(1 + \frac{\mathfrak{I}_S}{L})\right) \quad (7.43)$$

again, uniformly in $\omega \in \mathbb{R}$ and owing to (3.92). The leading term can be identified with the derivative of the dressed momentum of r -strings owing to (2.23). \blacksquare

Lemma 7.6. *Let $\Upsilon_{\text{red}} = \Upsilon \setminus \Upsilon^{(z)}$ and assume that hypothesis (3.36) holds. The matrix $\Xi_{\Upsilon_{\text{red}}}$ defined in (7.14) is invertible provided that L is large enough. Furthermore, one has*

$$\det[\Xi_{\Upsilon_{\text{red}}}] = \det[\text{id} + K] \cdot \left(1 + O\left(\frac{1}{L}\right)\right) \quad (7.44)$$

and

$$\det[\Xi_{\Lambda_b^{(\omega)}}] = \det[\text{id} + K] \cdot \left(1 + O\left(\frac{1}{L}\right)\right). \quad (7.45)$$

Proof—

Let $r \geq 1$. Then, for any function f holomorphic in a neighbourhood of \mathbb{R} it holds

$$\sum_{\alpha \in \Upsilon_{\text{red}}} \frac{K_{r,1}(\lambda - \alpha)f(\alpha)}{L\widehat{\xi}'_{\Upsilon}(\alpha)} = \int_{-q}^q f(s)K_{r,1}(\lambda - s) \cdot ds + \mathcal{O}_r[f](\lambda) \quad (7.46)$$

with

$$\begin{aligned} \mathcal{O}_r[f](\lambda) = & \sum_{\epsilon = \pm} \int_{\mathcal{C}^{(\epsilon)}} \frac{K_{r,1}(\lambda - s) \cdot (\widehat{u}_{\Upsilon}^{(\epsilon)})'(s)}{L\widehat{\xi}'_{\Upsilon}(s)} \cdot f(s) \cdot \frac{ds}{2i\pi} - \sum_{\substack{\alpha \in V \cup \Upsilon^{(p)} \\ \setminus \Upsilon^{(h)}}} \frac{K_{r,1}(\lambda - \alpha)f(\alpha)}{L\widehat{\xi}'_{\Upsilon}(\alpha)} \\ & + \int_{\mathbb{R}} K_{r,1}(\lambda - s)f(s)[\mathbf{1}_{[\widehat{q}_L; \widehat{q}_R]}(s) - \mathbf{1}_{I_q}(s)] \cdot ds \end{aligned} \quad (7.47)$$

and where $V = \{\mu_a^{(s)}\}_1^{n_{\text{sg}}}$. For $\delta > 0$ small enough, \mathcal{O}_r is a continuous linear operator on $L^\infty(\mathcal{S}_\delta(\mathbb{R}))$ and

$$\|\mathcal{O}_r[f]\|_{L^\infty(\mathcal{S}_\delta(\mathbb{R}))} \leq C \frac{n_{\text{sg}} + 1 + \mathfrak{V}/L}{L} \|f\|_{L^\infty(\mathcal{S}_\delta(\mathbb{R}))}. \quad (7.48)$$

Such estimates can be obtained as follows.

- The estimates (B.7) and (3.49) ensure that $\widehat{\xi}'_{\Upsilon} > c > 0$ for some constant c on the compact \mathcal{C} . Then, (B.14) allows one to bound the first term in (7.47).
- The integral appearing in the second line of (7.47) can be bounded thanks to (3.6).
- The discrete sum over $\Upsilon^{(p)} \setminus \Upsilon^{(h)}$ can be bounded by first, invoking (3.36) and then using (B.6), the expansion (3.49) and the fact that $\Upsilon^{(p)}$ and $\Upsilon^{(h)}$ are both bounded in L . The occurrence of $(1 + \mathfrak{V})/L$ then appears due to further simplifications for the massless modes.
- Finally, the sum over V is bounded by using the estimates (B.2) on $\widehat{\xi}'_{\Upsilon}(\alpha)$.

Expanding $\det[\Xi_{\Upsilon_{\text{red}}}]$ into a discrete Fredholm series and then replacing the discrete sums over the elements of $\Upsilon^{(\text{in})}$ one gets

$$\det[\Xi_{\Upsilon_{\text{red}}}] = \sum_{n \geq 0} \frac{1}{n!} \prod_{a=1}^n \left\{ \int_{\mathcal{C}} \frac{ds_a}{e^{2i\pi L\widehat{\xi}'_{\Upsilon}(s_a)} - 1} - \sum_{\substack{s_a \in V \cup \Upsilon^{(p)} \\ \setminus \Upsilon^{(h)}}} \frac{1}{L\widehat{\xi}'_{\Upsilon}(s_a)} \right\} \cdot \det_n[K(s_a - s_b)] \quad (7.49)$$

with $V = \{\mu_1^{(s)}, \dots, \mu_{n_{\text{sg}}}^{(s)}\}$. Upon a slight rewriting of the integrations, and upon interpreting the operator \mathcal{O}_1 as an operator on $L^2([\min(-q, \widehat{q}_L); \max(q, \widehat{q}_R)] \cup \mathcal{C}^{(+)} \cup \mathcal{C}^{(-)} \cup V \cup \Upsilon^{(p)} \cup \Upsilon^{(h)})$ one gets

$$\det[\Xi_{\Upsilon_{\text{red}}}] = \det[\text{id} + \mathbb{K} + \mathcal{O}_1]. \quad (7.50)$$

The operator \mathbb{K} appearing in the *rhs* of this equality corresponds to the injection of the operator K into the L^2 -space defined earlier. The determinant is well defined since, according to the criterion established in [11], the operators K and \mathcal{O}_1 are trace class: they acts on functions supported on a compact and have integral kernels that are of class C^1 . They are as well Hilbert-Schmidt. It is easily seen on the basis of

(7.48) that $|\text{tr}[O_1]| + \|O_1\|_2 = O(L^{-1})$ where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. Then, owing to the bound (6.40), one readily gets that

$$|\det[\text{id} + K + O_1] - \det[\text{id} + K]| \leq C \cdot \frac{1}{L} \quad (7.51)$$

so that (7.44) follows. The expansion (7.45) is obtained by similar handlings. \blacksquare

Lemma 7.7. *Under the hypotheses (3.36) and (3.37), it holds*

$$\widehat{R}(\lambda, \mu) = R(\lambda, \mu) + O\left(\frac{1}{L}\right) \quad \text{uniformly in } \lambda, \mu \in \mathbb{R} \quad (7.52)$$

and

$$\begin{aligned} \widehat{R}_{ab}^{(rs)}(c_a^{(r)}, c_b^{(s)}) &= K_{r,s}(c_a^{(r)} - c_b^{(s)}) - \int_{-q}^q K_{r,1}(c_a^{(r)} - t) K_{1,s}(t - c_b^{(s)}) \cdot dt \\ &\quad + \int_{-q}^q K_{r,1}(c_a^{(r)} - t) R(t, v) K_{1,s}(v - c_b^{(s)}) \cdot dt dv + O\left(\frac{1}{L}\right) \end{aligned} \quad (7.53)$$

Proof. The first step consists in characterising the discrete resolvent $\widehat{R}(\lambda, \mu)$ for $\lambda, \mu \in \mathbb{R}$. Since $\Xi_{Y_{\text{reg}}}$ is invertible for L large enough, the latter is well defined by means of (7.17), (7.18) and (7.19). In fact, starting from (7.19) and by using the summation identity (7.46), the discrete resolvent can be recast as

$$\widehat{R}(\lambda, \mu) = K(\lambda - \mu) - \int_{-q}^q K(\lambda - s) \widehat{R}(s, \mu) \cdot ds - O_1[\widehat{R}(*, \mu)](\lambda) \quad (7.54)$$

leading to

$$\widehat{R}(\lambda, \mu) = R(\lambda, \mu) - (\text{id} - R) \circ O_1[\widehat{R}(*, \mu)](\lambda). \quad (7.55)$$

Owing to (7.48) and to the continuity of $\text{id} - R$ one has the operator norm bound

$$\|(\text{id} - R) \circ O_1\|_{\mathcal{L}(L^\infty(\mathcal{S}_\delta(\mathbb{R})))} \leq \frac{C}{L} \quad (7.56)$$

and hence $\text{id} + (\text{id} - R) \circ O_1$ is invertible on $L^\infty(\mathcal{S}_\delta(\mathbb{R}))$. The Neumann series representation for this inverse then ensures that, in fact, $\widehat{R}(\lambda, \mu)$ is holomorphic on $\mathcal{S}_\delta(\mathbb{R}) \times \mathcal{S}_\delta(\mathbb{R})$ for some $\delta > 0$ small enough and that

$$\widehat{R}(\lambda, \mu) = R(\lambda, \mu) + O\left(\frac{1}{L}\right) \quad (7.57)$$

with a remainder that is uniform on $\mathcal{S}_\delta(\mathbb{R}) \times \mathcal{S}_\delta(\mathbb{R})$.

Starting from the expression (7.19) and then using the bounds (7.35), the fact that \widehat{R} is bounded on $Y \setminus Y^{(z)}$ by virtue of (7.57) and also the fact that $\widehat{\xi}_Y'(\alpha) > C > 0$ for $\alpha \in Y \setminus Y^{(z)}$, one gets

$$|\widehat{R}_{ab}^{(rs)}(c_a^{(r)}, c_b^{(s)}) - \widehat{R}_{r,s}(c_a^{(r)}, c_b^{(s)})| \leq C |Y \setminus Y^{(z)}|^2 \cdot \max\{\delta_{a,k}^{(r)}\} L^{2k} = O(L^{-\infty}) \quad \text{for } (r, a) \neq (s, b). \quad (7.58)$$

The function $\widehat{R}_{r,s}$ appearing above is expressed, owing to the summation identity (2.4), as

$$\widehat{R}_{r,s}(\lambda, \mu) = K_{r,s}(\lambda - \mu) - \sum_{\alpha \in \Upsilon \setminus \Upsilon'(z)} \frac{K_{r,1}(\lambda - \alpha) K_{1,s}(\alpha - \mu)}{L \widehat{\xi}'_{\Upsilon}(\alpha)} + \sum_{\alpha, \beta \in \Upsilon \setminus \Upsilon'(z)} \frac{K_{r,1}(\lambda - \alpha) \widehat{R}(\alpha, \beta) K_{1,s}(\beta - \mu)}{L \widehat{\xi}'_{\Upsilon}(\alpha) \cdot L \widehat{\xi}'_{\Upsilon}(\beta)}. \quad (7.59)$$

By using (7.46), one can recast $\widehat{R}_{r,s}(\lambda, \mu)$ in the form

$$\begin{aligned} \widehat{R}_{r,s}(\lambda, \mu) = & K_{r,s}(\lambda - \mu) - \int_{-q}^q K_{r,1}(\lambda - t) K_{1,s}(t - \mu) \cdot dt \\ & + \int_{-q}^q K_{r,1}(\lambda - t) R(t, v) K_{1,s}(v - \mu) \cdot dt dv + \mathfrak{R}_{r,s}[\widehat{R}](\lambda, \mu) \end{aligned} \quad (7.60)$$

where

$$\begin{aligned} \mathfrak{R}_{r,s}[\widehat{R}](\lambda, \mu) = & \int_{-q}^q \left\{ K_{r,1}(\lambda - t) O_s[\widehat{R}(t, *)](\mu) + O_r[\widehat{R}(*, t)](\lambda) K_{1,s}(t - \mu) \right\} \cdot dt \\ & - O_r[K_{1,s}(* - \mu)](\lambda) + O_r \otimes O_s[\widehat{R}(*, *)](\lambda, \mu) + \int_{-q}^q K_{r,1}(\lambda - t) \left\{ \widehat{R}(t, v) - R(t, v) \right\} K_{1,s}(v - \mu) \cdot dt dv. \end{aligned} \quad (7.61)$$

The estimates (7.57) on \widehat{R} and the continuity (7.48) of the operators O_s then allow one to conclude. \blacksquare

8 Conclusion

This paper addressed the problem of extracting the large-volume asymptotic behaviour of the form factors of local operators in the massless regime of the XXZ spin-1/2 chain in the case where the expectation value is taken between the ground state at finite magnetic field below the critical one and an excited state containing both, particle-hole excitations and bound states. The asymptotic expansion were obtained on rigorous grounds and provide an explicit control of the remainder uniformly in respect to a large-class of excited states built by the Bethe Ansatz. Such a precise control opens the possibility to extract rigorously, under mild assumptions,

- the large-distance and long-time asymptotic expansion of two-point functions of the model;
- the edge exponents and the associated non-universal prefactors characterising the singular behaviour of the space and time Fourier transforms of two-point function.

The results obtained in this paper thus provide one with a fundamental tool allowing one to deal with the dynamical properties of a model containing bound states and particle-hole excitations. I refer to the forthcoming paper for a better exposition of the problematic.

Acknowledgements

K.K.K. acknowledges support from a CNRS-installation grant. The author is indebted to F. G   hmann, N. Kitanine, J.M. Maillet, G. Niccoli for stimulating discussions.

A Asymptotic expansions of auxiliary integral transforms

A.1 Special functions of interest and auxiliary results

The Euler Gamma function satisfies to the uniform in $\Delta \geq 0$ and $z \in]\epsilon; +\infty[$, $\epsilon > 0$, estimate

$$\frac{\Gamma(z + \Delta)}{\Gamma(z)} = z^\Delta \left\{ 1 + O\left(\frac{\Delta}{z}\right) \right\} \quad (\text{A.1})$$

The Barnes function is a generalisation of the Gamma function. Its ratio admits the integral representation

$$(2\pi)^z \cdot \frac{G(1-z)}{G(1+z)} = \exp \left\{ \int_0^z \pi x \cot(\pi x) dx \right\} \quad (\text{A.2})$$

and it satisfies to the reflection identity

$$(-1)^{\frac{1}{2}\ell(\ell+1)} \frac{G(1-z-\ell)G(1+z)}{G(1+z+\ell)G(1-z)} = \left(\frac{\sin(\pi z)}{\pi} \right)^\ell \quad (\text{A.3})$$

Lemma A.1. [19] *Let U be a simply connected domain in \mathbb{C} and K be a compact in U . Given a holomorphic function f on U , denote by $N_K(f)$ the number of its zeroes, counted with their multiplicities and lying inside of K . Then, there exists $C > 0$ such that, for any f holomorphic on U , continuous on ∂U and having no zeroes on ∂U , one has the bound*

$$N_K(f) \leq C \frac{\ln \|f\|_{L^\infty(U)}}{\ln \|f\|_{L^\infty(K)}}. \quad (\text{A.4})$$

A.2 Uniform asymptotics of the Cauchy transform

For $v \in \{L, R\}$, let \mathbb{M}_v be the integral transform

$$\widehat{\mathbb{M}}_v[f](\omega) = - \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{f(s) - f(\omega)}{\tanh(s - \omega)} \cdot ds + i\pi \mathbf{1}_{\text{Int}(\mathcal{C})}(\omega) f(\omega) - f(\omega) \ln \left(\frac{\sigma_v \sinh(\omega - \widehat{q}_R)}{\sinh(\omega - \widehat{q}_L) [\widehat{\xi}_\Lambda(\omega) - \widehat{\xi}_\Lambda(\widehat{q}_v)]^{\sigma_v}} \right) \quad (\text{A.5})$$

where σ_v is as given in (3.48).

For the purpose of this appendix, given $\Omega \in \{\Upsilon, \Lambda_b^{(a)}\}$, I adopt the notations

$$\varsigma_v^{(\Omega)}(\omega) = \tau_v + \delta_{\Omega; \Upsilon} \left(\varkappa_v - \widehat{F}_{\text{reg}}(\omega) \right) \Big|_{\alpha_\Lambda=0} + \delta_{\Omega; \Lambda_b^{(a)}} \widehat{\mathfrak{f}}(\omega) \quad \text{and, for short,} \quad \varsigma_v^{(\Omega)} = \varsigma_v^{(\Omega)}(\widehat{q}_v) \quad (\text{A.6})$$

where the regular part \widehat{F}_{reg} of the shift function has been defined in (3.45),

$$\widehat{\mathfrak{f}}(\omega) = L \left(\widehat{\xi}_{\Lambda^{(a)}}^{(b)}(\omega) - \widehat{\xi}_\Lambda(\omega) \right) \quad \text{and} \quad \widehat{\mathfrak{f}}_v = \widehat{\mathfrak{f}}(\widehat{q}_v). \quad (\text{A.7})$$

Proposition A.2. *Given any $\delta > 0$ small enough, the Cauchy transform $C_{\mathcal{C}}[\widehat{z}](\omega)$ admits the asymptotic expansion*

$$C_{\mathcal{C}}[\widehat{z}](\omega) = 2i\pi C_{\mathcal{C}^{(+)}}[\widehat{F}_{\text{reg}}^{(b)}](\omega) + O\left(\frac{1}{L}\right) \quad \text{for } \omega \in \{\mathbb{C} \setminus \mathcal{D}_{\widehat{q}_L, \delta} \cup \mathcal{D}_{\widehat{q}_R, \delta}\} / \{i\pi\mathbb{Z}\}. \quad (\text{A.8})$$

When $\omega \in \mathcal{D}_{\widehat{q}_v, \delta}$ with $v \in \{L, R\}$, the large- L asymptotic expansion takes the form

$$\begin{aligned} C_{\mathcal{C}}[\widehat{z}](\omega) &= \widehat{M}_v[\widehat{F}_{\text{reg}}^{(b)}](\omega) + \sigma_v \left(\widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v) - \kappa_v - \widehat{F}_{\text{reg}}^{(b)}(\omega) \right) \cdot \ln \left[\pm \sigma_v L \cdot \left(\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(\omega) \right) \right] \\ &\quad + \sigma_v \widehat{F}_{\text{reg}}^{(b)}(\omega) \ln L \mp \ln \Gamma \left(\frac{\frac{1}{2} \pm \sigma_v \left(L \cdot [\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(\omega)] - s_v^{(\Upsilon)} \right)}{\frac{1}{2} \pm \sigma_v \left(L \cdot [\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(\omega)] - s_v^{(\Lambda_b^{(a)})} \right)} \right) + O\left(\frac{\ln L}{L}\right). \end{aligned} \quad (\text{A.9})$$

In each of these local asymptotics, the remainder is uniform in the whole region where the expansion holds. Also, the $+$ sign corresponds to $\omega \in \text{Int}(\mathcal{C})/\{i\pi\mathbb{Z}\}$ while the $-$ sign corresponds to $\omega \in \text{Ext}(\mathcal{C})/\{i\pi\mathbb{Z}\}$.

Proof—

By using the notation (3.72), one can decompose the Cauchy transform as

$$C_{\mathcal{C}}[\widehat{u}_{\Omega}](\omega) = -2i\pi L C_{\mathcal{C}^{(+)}}[\widehat{\xi}_{\Omega_{\text{reg}}}](\omega) + \sum_{\epsilon=\pm} C_{\mathcal{C}^{(\epsilon)}}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega) + \sum_{\epsilon=\pm} C_{\mathcal{C}^{(\epsilon)}}[u_{\Omega}^{(\epsilon)}](\omega) \quad (\text{A.10})$$

with

$$u_{\Omega}^{(\epsilon)} = \widehat{u}_{\Omega}^{(\epsilon)} - \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)} - 2i\pi L \delta_{\epsilon,+} \widehat{\xi}_{\Omega_{\text{sing}}}. \quad (\text{A.11})$$

Note that $u_{\Lambda_b^{(a)}}^{(\epsilon)} = 0$, so that the last term in (A.10) is only relevant when $\Omega = \Upsilon$. Then, however, this term only generates uniform $O(L^{-\infty})$ corrections. Indeed, let \ln_{ϵ} be some determination of the logarithm such that

- i) $\ln_{\epsilon}(s - \omega)$ is holomorphic on $\overline{\mathbb{H}}^{\epsilon}$;
- ii) $\ln_{+}(\widehat{q}_v - \omega) - \ln_{-}(\widehat{q}_v - \omega) = 2i\pi n_v(\omega)$ with $n_v(\omega)$ bounded in L , this uniformly in $\omega \in \{\mathbb{C} \setminus \mathcal{D}_{\widehat{q}_L, \delta} \cup \mathcal{D}_{\widehat{q}_R, \delta}\} / \{i\pi\mathbb{Z}\}$;
- iii) $\|\ln_{\epsilon}(* - \omega)\|_{L^1(\mathcal{C}^{(\epsilon)})} \leq C$.

Then, it holds

$$\begin{aligned} \sum_{\epsilon=\pm} C_{\mathcal{C}^{(\epsilon)}}[u_{\Upsilon}^{(\epsilon)}](\omega) &= \sum_{\epsilon=\pm} \frac{\epsilon}{2i\pi} \left\{ u_{\Upsilon}^{(\epsilon)}(\widehat{q}_L) \cdot \ln_{\epsilon}(\widehat{q}_L - \omega) - u_{\Upsilon}^{(\epsilon)}(\widehat{q}_R) \cdot \ln_{\epsilon}(\widehat{q}_R - \omega) \right\} \\ &\quad - \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} (u_{\Upsilon}^{(\epsilon)})'(s) \ln_{\epsilon}(s - \omega) \cdot \frac{ds}{2i\pi}. \end{aligned} \quad (\text{A.12})$$

It is readily checked, owing to (B.7) and (3.60), that $u_{\Upsilon}^{(+)}(\widehat{q}_v) = u_{\Upsilon}^{(-)}(\widehat{q}_v)$. This ensures that the logarithmic singularities at the boundaries cancel out and, due to the other properties of $\ln_{\epsilon}(s - \omega)$, one gets

$$\left\| \sum_{\epsilon=\pm} C_{\mathcal{C}^{(\epsilon)}}[u_{\Upsilon}^{(\epsilon)}] \right\|_{L^{\infty}(K)} \leq C' \sum_{\epsilon=\pm} \|u_{\Upsilon}^{(\epsilon)}\|_{W_1^{\infty}(\mathcal{C}^{(\epsilon)})} = O(L^{-\infty}) \quad (\text{A.13})$$

due to (B.7) and (B.13), where K is some sufficiently narrow compact neighbourhood of \mathcal{C} in \mathbb{C} . One obtains similar bounds on $\{\mathbb{C} \setminus K\}/\{i\pi\mathbb{Z}\}$ this time by straightforward bounds since $\tanh|s-z|$ is uniformly bounded from below for $s \in \mathcal{C}/\{i\pi\mathbb{Z}\}$ and $z \in \{\mathbb{C} \setminus K\}/\{i\pi\mathbb{Z}\}$.

If $\omega \in \{\mathbb{C} \setminus \mathcal{D}_{\widehat{q}_L, \delta} \cup \mathcal{D}_{\widehat{q}_R, \delta}\}/\{i\pi\mathbb{Z}\}$, then, if necessary, deforming the part of \mathcal{C} that is uniformly away from \widehat{q}_v so as to make the distance of ω to \mathcal{C} finite, the bound (B.14) leads to

$$\left| \mathbb{C}_{\mathcal{C}^{(\epsilon)}}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega) \right| = O(L^{-1}). \quad (\text{A.14})$$

All together, these bounds yield (A.8).

The treatment of $\omega \in \mathcal{D}_{\widehat{q}_v, \delta}$ needs more care. The bound (A.13) ensures that the last term in (A.10) still produces $O(L^{-\infty})$ corrections in that case. The first term in (A.10) can be decomposed as

$$\begin{aligned} -2i\pi L \mathbb{C}_{\mathcal{C}^{(+)}}[\widehat{\xi}_{\Omega_{\text{reg}}}](\omega) &= L \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{\widehat{\xi}_{\Omega_{\text{reg}}}(s) - \widehat{\xi}_{\Omega_{\text{reg}}}(\omega)}{\tanh(s - \omega)} \cdot ds \\ &\quad + L \widehat{\xi}_{\Omega_{\text{reg}}}(\omega) \left[\ln \left(\frac{\sinh(\widehat{q}_R - \omega)}{\sinh(\widehat{q}_L - \omega)} \right) - 2i\pi \mathbf{1}_{\text{Int}(\mathcal{C}) \cap \mathbb{H}^+}(\omega) \right]. \end{aligned} \quad (\text{A.15})$$

In order to estimate the behaviour of $\mathbb{C}_{\mathcal{C}^{(\epsilon)}}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega)$, it is useful to decompose the contour \mathcal{C} into portions neighbouring the endpoints \widehat{q}_v and portions that are uniformly away from \mathbb{R} . For this purpose, one introduces the intervals

$$J_{\delta}^{(L)} = \frac{1/2 + \tau_L}{L} + [i\delta; -i\delta] \quad \text{and} \quad J_{\delta}^{(R)} = \frac{1/2 + |\Lambda| + \tau_R}{L} + [-i\delta; i\delta] \quad (\text{A.16})$$

which then allow one to define the sub-sets of \mathcal{C}

$$\mathcal{C}^{(\text{out})} = \widehat{\xi}_{\Lambda}^{-1}(\widehat{\Gamma} \setminus \{J_{\delta}^{(L)} \cup J_{\delta}^{(R)}\}), \quad \mathcal{C}^{(\text{out}; \pm)} = \mathcal{C}^{(\text{out})} \cap \mathbb{H}^{\pm} \quad (\text{A.17})$$

and

$$\mathcal{C}^{(v)} = \widehat{\xi}_{\Lambda}^{-1}(J_{\delta}^{(v)}) \quad , \quad \mathcal{C}^{(v; \epsilon)} = \mathcal{C}^{(v)} \cap \mathbb{H}^{\epsilon}. \quad (\text{A.18})$$

Let $\bar{v} = L$ if $v = R$ and $\bar{v} = R$ if $v = L$. Then one decomposes the Cauchy transforms as

$$\begin{aligned} \sum_{\epsilon=\pm} \mathbb{C}_{\mathcal{C}^{(\epsilon)}}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega) &= \sum_{\epsilon=\pm} \mathbb{C}_{\mathcal{C}^{(\text{out}; \epsilon)}}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega) + \sum_{\epsilon=\pm} \mathbb{C}_{\mathcal{C}^{(\bar{v}; \epsilon)}}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega) \\ &\quad + \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(v; \epsilon)}} \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \left\{ \coth(s - \omega) - \frac{\widehat{\xi}'_{\Lambda}(s)}{\widehat{\xi}_{\Lambda}(s) - \widehat{\xi}_{\Lambda}(\omega)} \right\} \cdot \frac{ds}{2i\pi} + \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(v; \epsilon)}} \frac{\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \cdot \widehat{\xi}'_{\Lambda}(s)}{\widehat{\xi}_{\Lambda}(s) - \widehat{\xi}_{\Lambda}(\omega)} \cdot \frac{ds}{2i\pi}. \end{aligned}$$

Since the $\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}$ independent part of the integrand is bounded in the case of the first three terms, the estimate (B.14) ensures that these are a $O(L^{-1})$. In its turn, upon the change of variables

$$s_t = \widehat{\xi}_{\Lambda}^{-1}(\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \frac{\sigma_v t}{2i\pi L}) \quad \text{and after setting} \quad a = 2i\pi\sigma_v L(\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(\omega)) \quad (\text{A.19})$$

the last term can be recast in the form

$$\sum_{\epsilon=\pm} \int_{\mathcal{C}^{(v; \epsilon)}} \frac{\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \cdot \widehat{\xi}'_{\Lambda}(s)}{\widehat{\xi}_{\Lambda}(s) - \widehat{\xi}_{\Lambda}(\omega)} \frac{ds}{2i\pi} = \int_{-2\pi\delta L}^{2\pi\delta L} \frac{\ln(1 + e^{-|t| + 2i\pi\zeta_v^{(\Omega)}(s_t)\sigma_v \text{sgn}(t)})}{t - a} \cdot \frac{dt}{2i\pi}. \quad (\text{A.20})$$

Note that $a \in \mathbb{H}^+$ if $\omega \in \text{Int}(\mathcal{C})/\{i\pi\mathbb{Z}\}$ and $a \in \mathbb{H}^-$ if $\omega \in \text{Ext}(\mathcal{C})/\{i\pi\mathbb{Z}\}$. One can expand the s_t dependent part into a series in t . By using bounds as in (B.19) and integrating by parts, since $\|\widehat{F}_{\text{reg}}\|_{L^\infty(S_\delta(\mathbb{R}))} < Cn_{\text{tot}}^{(\text{msv})}$ is bounded in virtue of (3.52), one gets that the t -dependent part of the expansion produces $O(\ln L/L)$ corrections. Once the replacement $s_t \leftrightarrow s_0$ is made one can extend the integration to \mathbb{R} , this for the price of $O(L^{-\infty})$ corrections. The resulting integral can then be computed by means of Lemma C.1. All of this yields:

$$\begin{aligned} \sum_{\epsilon=\pm} c_{\mathcal{C}(\epsilon)}[\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}](\omega) &= \mp \ln \Gamma\left(\frac{1}{2} \pm \sigma_v \left[L(\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)) - s_v^{(\Omega)} \right] \right) \\ &\quad + \sigma_v \left[L(\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)) - s_v^{(\Omega)} \right] \cdot \ln \left[\pm \sigma_v L(\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)) \right] \\ &\quad \pm \ln \sqrt{2\pi} - \sigma_v L(\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)) + O\left(\frac{\ln L}{L}\right). \end{aligned} \quad (\text{A.21})$$

In order to obtain the claimed form for the Cauchy transform, one needs to observe that

$$\ln\left(\frac{\sinh(\omega - \widehat{q}_R)}{\sinh(\omega - \widehat{q}_L)}\right) = \ln\left(\frac{\sigma_v \sinh(\omega - \widehat{q}_R)}{\sinh(\omega - \widehat{q}_L)[\widehat{\xi}_\Lambda(\omega) - \widehat{\xi}_\Lambda(\widehat{q}_v)]^{\sigma_v}}\right) + \sigma_v \ln(\sigma_v[\widehat{\xi}_\Lambda(\omega) - \widehat{\xi}_\Lambda(\widehat{q}_v)]) \quad (\text{A.22})$$

and that, for $\omega \in \text{Int}(\mathcal{C})$,

$$\ln(\sigma_v[\widehat{\xi}_\Lambda(\omega) - \widehat{\xi}_\Lambda(\widehat{q}_v)]) = \ln(\sigma_v[\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)]) + i\pi\sigma_v \text{sgn}(\Im(\omega)). \quad (\text{A.23})$$

The rest is straightforward algebra. ■

Proposition A.2 provides one with the large- L asymptotics of the $\mathcal{L}_{\mathcal{C}}$ and $\widetilde{\mathcal{L}}_{\mathcal{C}}$ transforms.

Corollary A.3. *Given any $\delta > 0$ small enough, given $\omega \in \{\mathbb{C} \setminus \mathcal{D}_{\widehat{q}_L, \delta} \cup \mathcal{D}_{\widehat{q}_R, \delta}\}/\{i\pi\mathbb{Z}\}$ it holds*

$$e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\omega)} = e^{-2i\pi c_{\mathcal{C}(+) }[F^{(\omega)}](\omega)} \cdot \frac{\sinh^{\kappa_L}(\omega - \widehat{q}_L)}{\sinh^{\kappa_R}(\omega - \widehat{q}_R)} \left(1 + O\left(\frac{1}{L}\right)\right). \quad (\text{A.24})$$

For $\omega \in \mathcal{D}_{\widehat{q}_v, \eta}$ with $v \in \{L, R\}$ one has

$$\begin{aligned} e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\omega)} &= \frac{e^{-\widehat{M}_v[\widehat{F}_{\text{reg}}^{(b)} - \kappa_v](\omega)}}{L^{\sigma_v(\widehat{F}_{\text{reg}}^{(b)}(\omega) - \kappa_v)}} \left\{ \pm \sigma_v L \cdot [\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)] \right\}^{-\sigma_v(\widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v) - \widehat{F}_{\text{reg}}^{(b)}(\omega))} \\ &\quad \times [\sinh(\omega - \widehat{q}_{\overline{v}})]^{\kappa_L - \kappa_R} \cdot \Gamma^{\pm 1} \left(\begin{array}{c} \frac{1}{2} \pm \sigma_v \left(L \cdot [\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)] - s_v^{(\Upsilon)} \right) \\ \frac{1}{2} \pm \sigma_v \left(L \cdot [\widehat{\xi}_\Lambda(\widehat{q}_v) - \widehat{\xi}_\Lambda(\omega)] - s_v^{(\Lambda_b^{(a)})} \right) \end{array} \right) \cdot \left(1 + O\left(\frac{\ln L}{L}\right)\right). \end{aligned} \quad (\text{A.25})$$

Above, $\overline{v} = L$ if $v = R$ and viceversa and one should take $+$ if $\omega \in \text{Int}(\mathcal{C})/\{i\pi\mathbb{Z}\}$ and $-$ otherwise. Further, one has

$$e^{\widetilde{\mathcal{L}}_{\mathcal{C}}[\widehat{z}](\omega)} = (-1)^{\kappa_L - \kappa_R} \cdot e^{\mathcal{L}_{\mathcal{C}}[\widehat{z}](\omega)}. \quad (\text{A.26})$$

A.3 Asymptotics of the \mathcal{A}_0 transform

The uniform asymptotics of the Cauchy transform $c_{\mathcal{C}}[\widehat{z}](\omega)$ allow one to determine the ones of $\mathcal{A}_0[\widehat{z}, \widehat{z}]$ which was introduced in (5.4). Prior to stating the result, I introduce the functional

$$\widehat{\mathcal{T}}[f] = \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{f'(s)f(t) - f'(t)f(s)}{2 \tanh(s - t)} dt ds + \sum_{v \in \{L, R\}} \sigma_v(f(\widehat{q}_v) - \kappa_v) \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{f(s) - f(\widehat{q}_v)}{\tanh(s - \widehat{q}_v)} ds. \quad (\text{A.27})$$

Lemma A.4. Let $\widehat{\mathbf{f}}_v$ be as defined in (5.45). Then, it holds

$$\begin{aligned} \mathcal{A}_0[\widehat{z}, \widehat{z}] &= \widehat{\mathcal{T}}[\widehat{F}_{\text{reg}}^{(b)}] + \sum_{v \in \{L, R\}} \left\{ -i\sigma_v \frac{\pi}{2} (\widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v))^2 + \widehat{\mathbf{f}}_v^{(b)} \widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v) \ln(L \sinh(\widehat{q}_R - \widehat{q}_L) \widehat{\xi}'_\Lambda(\widehat{q}_v)) \right. \\ &\quad \left. + i\pi\sigma_v \mathcal{K}_v \widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_v) + \ln G(1 - \widehat{\mathbf{f}}_v^{(b)}, 1 + \widehat{\mathbf{f}}_v^{(b)}) + \sigma_v \mathcal{K}_v \ln \Gamma\left(\frac{\frac{1}{2} + \sigma_v(\tau_v + \widehat{\mathbf{f}}_v)}{\frac{1}{2} + \sigma_v(\tau_v + \widehat{\mathbf{f}}_{v|\alpha_\Lambda=0})}\right) \right\} + O\left(\frac{\ln L}{L}\right) \end{aligned} \quad (\text{A.28})$$

Proof —

One has the decomposition $\mathcal{A}_0[\widehat{z}, \widehat{z}] = \sum_{a=1}^3 \mathcal{A}_0^{(a)}[\widehat{z}, \widehat{z}]$ with

$$\mathcal{A}_0^{(1)}[\widehat{z}, \widehat{z}] = - \int_{\mathcal{C}^{(+)}} (\widehat{F}_{\text{reg}}^{(b)}(t))' \cdot \mathcal{C}_{\mathcal{C};+}[\widehat{z}](t) \cdot dt = \int_{\mathcal{C}} \widehat{z}(s) \cdot \mathcal{C}_{\mathcal{C}^{(+);-}}[(\widehat{F}_{\text{reg}}^{(b)})'](s) ds \quad (\text{A.29})$$

$$\mathcal{A}_0^{(2)}[\widehat{z}, \widehat{z}] = - \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} (\widehat{u}_{\Gamma_{\text{reg}}}^{(\epsilon)} - \widehat{u}_{\Lambda_b^{(a)}}^{(\epsilon)})'(t) \cdot \mathcal{C}_{\mathcal{C};+}[\widehat{z}](t) \cdot \frac{dt}{2i\pi} \quad (\text{A.30})$$

$$\mathcal{A}_0^{(3)}[\widehat{z}, \widehat{z}] = - \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} (\widehat{u}_{\Gamma}^{(\epsilon)} - \widehat{u}_{\Gamma_{\text{reg}}}^{(\epsilon)} - 2i\pi\delta_{\epsilon;+} L \widehat{\xi}_{\Gamma_{\text{sing}}}')(t) \cdot \mathcal{C}_{\mathcal{C};+}[\widehat{z}](t) \cdot \frac{dt}{2i\pi}. \quad (\text{A.31})$$

We remind that $\mathcal{C}_{\mathcal{C};+}$ refers to the $+$ boundary value of the Cauchy transform on \mathcal{C} . The estimate $\mathcal{A}_0^{(3)}[\widehat{z}, \widehat{z}] = O(L^{-\infty})$ can be inferred from the bound

$$\|\mathcal{C}_{\mathcal{C};+}[\widehat{z}]\|_{L^1(\mathcal{C})} \leq C \ln L \quad (\text{A.32})$$

which is a consequence of Proposition A.2 and of the bounds (B.13)-(B.7).

Next, one recasts $\mathcal{A}_0^{(1)}[\widehat{z}, \widehat{z}]$ as

$$\begin{aligned} \mathcal{A}_0^{(1)}[\widehat{z}, \widehat{z}] &= 2i\pi \int_{\mathcal{C}^{(+)}} \widehat{F}_{\text{reg}}^{(b)}(s) \cdot \mathcal{C}_{\mathcal{C}^{(+);-}}[(\widehat{F}_{\text{reg}}^{(b)})'](s) \cdot ds \\ &\quad + \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\epsilon)}} (\widehat{u}_{\Gamma}^{(\epsilon)}(s) - \widehat{u}_{\Lambda_b^{(a)}}^{(\epsilon)}(s) - 2i\pi\delta_{\epsilon;+} L \widehat{\xi}_{\Gamma_{\text{sing}}}(s)) \cdot \mathcal{C}_{\mathcal{C}^{(+);-}}[(\widehat{F}_{\text{reg}}^{(b)})'](s) \cdot ds. \end{aligned} \quad (\text{A.33})$$

Upon observing that the Cauchy transform has logarithmic singularities at the edges \widehat{q}_v and invoking the bounds (B.7), (B.15) one infers that the second line produces at most $O(L^{-1} \cdot \ln L)$ contributions. The first line can be estimated by deforming the contour $\mathcal{C}^{(+)}$ to $[\widehat{q}_R; \widehat{q}_L]$ and then symmetrising the integral. All-in-all, one obtains

$$\mathcal{A}_0^{(1)}[\widehat{z}, \widehat{z}] = \int_{\widehat{q}_L}^{\widehat{q}_R} \frac{(\widehat{F}_{\text{reg}}^{(b)})'(s) \widehat{F}_{\text{reg}}^{(b)}(t) - (\widehat{F}_{\text{reg}}^{(b)})'(t) \widehat{F}_{\text{reg}}^{(b)}(s)}{2 \tanh(s-t)} dt ds + i\frac{\pi}{2} \sum_{v \in \{L, R\}} \sigma_v (\widehat{F}_{\text{reg}}^{(b)})^2(\widehat{q}_v) + O\left(\frac{\ln L}{L}\right). \quad (\text{A.34})$$

Finally, one has

$$\mathcal{A}_0^{(2)}[\widehat{z}, \widehat{z}] = \sum_{v \in \{L, R\}} \mathcal{A}_{0;v}^{(2)}[\widehat{z}, \widehat{z}] - \sum_{\epsilon=\pm} \int_{\mathcal{C}^{(\text{out};\epsilon)}} (\widehat{u}_{\Gamma_{\text{reg}}}^{(\epsilon)} - \widehat{u}_{\Lambda_b^{(a)}}^{(\epsilon)})'(t) \cdot \mathcal{C}_{\mathcal{C};+}[\widehat{z}](t) \cdot \frac{dt}{2i\pi}. \quad (\text{A.35})$$

The last term produces $O(L^{-\infty})$ corrections while, upon implementing the change of variables (A.19) and inserting the local asymptotics of the Cauchy transform, one gets

$$\begin{aligned} \mathcal{A}_{0;\nu}^{(2)}[\widehat{z}, \widehat{z}] &= - \int_{-2\pi L\delta}^{2\pi L\delta} \left\{ \frac{\operatorname{sgn}(t) \left(1 - (\widehat{F}_{\text{reg}}^{(b)})'(s_t) / (L \widehat{\xi}'_{\Lambda}(s_t)) \right)}{1 + e^{|t| - 2i\pi \operatorname{sgn}(t) \sigma_{\nu} \widehat{S}_{\nu}^{(\Upsilon)}(s_t)}} - \frac{\operatorname{sgn}(t)}{1 + e^{|t| - 2i\pi \operatorname{sgn}(t) \sigma_{\nu} \widehat{S}_{\nu}^{(\Lambda_b^{(a)})}(s_t)}} \right\} \\ &\times \left\{ \widehat{\mathbf{M}}_{\nu}[\widehat{F}_{\text{reg}}^{(b)}](s_t) + \sigma_{\nu} \widehat{F}_{\text{reg}}^{(b)}(s_t) \ln L + \ln \Gamma \left(\frac{\frac{1}{2} + \frac{t-a}{2i\pi}}{\frac{1}{2} + \frac{t-a-b}{2i\pi}} \right) \right. \\ &\quad \left. + \sigma_{\nu} \left(\widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_{\nu}) - \kappa_{\nu} - \widehat{F}_{\text{reg}}^{(b)}(s_t) \right) \cdot \ln[-i(t/2\pi + i0^+)] \right\} \cdot \frac{dt}{2i\pi} + O\left(\frac{\ln L}{L}\right). \quad (\text{A.36}) \end{aligned}$$

The form of the remainder follows from (B.14) and I have set $a = 2i\pi\sigma_{\nu}(\tau_{\nu} + \widehat{\mathbf{f}}_{\nu})$ and $b = 2i\pi\sigma_{\nu}\widehat{\mathbf{f}}_{\nu}^{(b)}$. The s_t dependent part can, again, be replaced by s_0 up to $O(L^{-1} \ln L)$ correction and then one can extend the integration to \mathbb{R} up to exponentially small corrections. One gets

$$\begin{aligned} \mathcal{A}_{0;\nu}^{(2)}[\widehat{z}, \widehat{z}] &= - \int_{\mathbb{R}} \left\{ \frac{\operatorname{sgn}(t)}{1 + e^{|t| - \operatorname{sgn}(t)a}} - \frac{\operatorname{sgn}(t)}{1 + e^{|t| - \operatorname{sgn}(t)(a+b)}} \right\} \left\{ \widehat{\mathbf{M}}_{\nu}[\widehat{F}_{\text{reg}}^{(b)}](\widehat{q}_{\nu}) + \sigma_{\nu} \widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_{\nu}) \ln L \right. \\ &\quad \left. - \sigma_{\nu} \kappa_{\nu} \cdot \ln[-i(t/2\pi + i0^+)] + \ln \Gamma \left(\frac{\frac{1}{2} + \frac{t-a}{2i\pi}}{\frac{1}{2} + \frac{t-a-b}{2i\pi}} \right) \right\} \cdot \frac{dt}{2i\pi} + O\left(\frac{\ln L}{L}\right). \quad (\text{A.37}) \end{aligned}$$

The resulting integrals can be computed by means of Lemma C.2, hence leading to

$$\begin{aligned} \mathcal{A}_{0;\nu}^{(2)}[\widehat{z}, \widehat{z}] &= \sigma_{\nu} \widehat{\mathbf{f}}_{\nu}^{(b)} \widehat{\mathbf{M}}_{\nu}[\widehat{F}_{\text{reg}}^{(b)}](\widehat{q}_{\nu}) + \widehat{\mathbf{f}}_{\nu}^{(b)} \widehat{F}_{\text{reg}}^{(b)}(\widehat{q}_{\nu}) \ln L \\ &\quad + \ln G(1 - \widehat{\mathbf{f}}_{\nu}^{(b)}, 1 + \widehat{\mathbf{f}}_{\nu}^{(b)}) + \sigma_{\nu} \kappa_{\nu} \ln \Gamma \left(\frac{\frac{1}{2} + \sigma_{\nu}(\tau_{\nu} + \widehat{\mathbf{f}}_{\nu})}{\frac{1}{2} + \sigma_{\nu}(\tau_{\nu} + \widehat{\mathbf{f}}_{\nu|\alpha_{\Lambda}=0})} \right) + O\left(\frac{\ln L}{L}\right). \quad (\text{A.38}) \end{aligned}$$

Upon straightforward algebra, one gets the claim. ■

B A few auxiliary bounds

B.1 The singular counting function and singular roots

Lemma B.1. *Assume that the algebraic spacing between string centres holds (3.37) and also the real/singular root spacing (3.36). Then, one has*

$$|(\widehat{\xi}_{\Upsilon^{(a)}})'(\mu_a^{(s)})| = O(1) \quad (\text{B.1})$$

and

$$\widehat{\xi}_{\Upsilon}'(\mu_a^{(s)}) = \frac{1}{L} K(\mu_a^{(s)} - \Re(\beta_a^{(s)}) | \Im(\beta_a^{(s)})) \cdot (1 + O(L^{-\infty})) \quad \text{so that} \quad |\widehat{\xi}_{\Upsilon}'(\mu_a^{(s)})| \geq CL^k \quad (\text{B.2})$$

for any $k \in \mathbb{N}$ and $C > 0$.

Proof—

One has, by construction, that

$$\left(\widehat{\xi}_{\mathbf{r}_{\text{reg}}^{(a)}}\right)'(\mu_a^{(s)}) = \widehat{\xi}'_{\mathbf{r}_{\text{reg}}}(\mu_a^{(s)}) + \frac{1}{L} \sum_{\beta + i\zeta \in Z^{(s)}, \beta \neq \beta_a^{(s)}} K(\mu_a^{(s)} - \Re(\beta) \mid \Im(\beta)). \quad (\text{B.3})$$

The first term is clearly an $O(1)$. As for the terms arising in the sum, since $\beta \neq \beta_a^{(s)}$, one has, due to (3.37), that

$$\min_{\beta \neq \beta_a^{(s)}} (|\mu_a^{(s)} - \beta|, |\mu_a^{(s)} - \beta^*|) > C \cdot L^{-\kappa} \quad \text{so that} \quad K(\mu_a^{(s)} - \Re(\beta) \mid \Im(\beta)) = O(L^{-\infty}) \quad (\text{B.4})$$

where the bound also follows from the estimate $\Im(\beta) = O(L^{-\infty})$. This result will then entail (B.2) as soon as one has the lower bound on $|K(\mu_a^{(s)} - \Re(\beta_a^{(s)}) \mid \Im(\beta_a^{(s)}))|$. Recall that $\mu_a^{(s)}$ belongs to a disk of radius $C|\Im(\beta_a^{(s)})|^{1-\nu}$ around $\beta_a^{(s)}$ for some L independent $C > 0$, c.f. (3.39). Thus, one has that

$$\left|K(\mu_a^{(s)} - \Re(\beta_a^{(s)}) \mid \Im(\beta_a^{(s)}))\right| \geq \frac{C'|\Im(\beta_a^{(s)})|}{|\mu_a^{(s)} - \beta_a^{(s)}|^2} \geq \frac{C''}{|\Im(\beta_a^{(s)})|^{1-2\nu}} \quad (\text{B.5})$$

and the claim follows since $\nu < 1/2$. ■

Lemma B.2. Assume (3.37)-(3.36). Then, given $k \in \mathbb{N}$, one has

$$\|\widehat{\xi}_{\mathbf{r}_{\text{sing}}}\|_{L^\infty(\mathcal{Z}^{(s)})} = O(n_{\text{sg}} L^{-\infty}) \quad (\text{B.6})$$

with

$$\mathcal{Z}^{(s)} = \left\{ z \in \mathcal{S}_\delta(\mathbb{R}) \quad : \quad d\left(z, c_a^{(r)} + \delta_{a, \frac{r+1}{2}}^{(r)}\right) > C|\Im(\delta_{a, \frac{r-1}{2}}^{(r)})|^\nu \quad r \text{ odd}, \quad \begin{array}{l} r = 3, \dots, p_{\max} \\ a = 1, \dots, n_r^{(z)} \end{array} \right\}.$$

In particular, one has

$$\|\widehat{\xi}_{\mathbf{r}_{\text{sing}}}\|_{W_k^\infty(\mathcal{C})} = O(n_{\text{sg}} L^{-\infty}). \quad (\text{B.7})$$

Furthermore, given $\widehat{\xi}_{\mathbf{r}_{\text{sing}}^{(a)}}$ as in (3.125) and $R > 0$, one has

$$\|\widehat{\xi}_{\mathbf{r}_{\text{sing}}^{(a)}}\|_{L^\infty(\mathcal{D}_a)} = O(n_{\text{sg}} L^{-\infty}) \quad \text{with} \quad \mathcal{D}_a = \mathcal{D}_{\beta_a^{(s)}, R|\Im(\beta_a^{(s)})|} \quad (\text{B.8})$$

where $\mathcal{Z}^{(s)}(\beta_a^{(s)})$ is defined analogously to $\mathcal{Z}^{(s)}$ with the exception that the central roots $\mu_a^{(s)}$ associated with $\beta_a^{(s)}$ is absent from the constraint on the lower bound on the distances.

Proof—

The representation

$$\widehat{\xi}_{\mathbf{r}_{\text{sing}}}(\omega) = \frac{1}{2i\pi L} \sum_{\substack{\beta + i\zeta \\ \in Z^{(s)}}} \ln \left[\cosh(\beta - \beta^*) - \coth(\beta^* - \omega) \sinh(\beta - \beta^*) \right] \quad (\text{B.9})$$

which follows readily from (3.25) leads to the bounds

$$\left| \widehat{\xi}_{\Upsilon_{\text{sing}}}(\omega) \right| \leq C \cdot \frac{\#Z^{(s)}}{L} \max_{\substack{\beta+i\zeta \\ \in Z^{(s)}}} \left| \frac{\beta - \beta^*}{\omega - \beta^*} \right|. \quad (\text{B.10})$$

Then, by using (3.39), (B.6) follows, just as (B.7) for $k = 0$ since $d(\beta_a^{(s)}, \mathcal{C}) \geq C/L$. Further, with $\text{sinhc}(x) = \sinh(x)/x$, for $k \geq 1$, one has

$$\widehat{\xi}_{\Upsilon_{\text{sing}}}^{(k)}(\omega) = \frac{1}{2i\pi L} \sum_{\substack{\beta+i\zeta \\ \in Z^{(s)}}} \ln^{(k)} \left(\frac{\text{sinhc}(\beta - \omega)}{\text{sinhc}(\beta^* - \omega)} \right) + \frac{(-1)^{k-1}(k-1)!}{2i\pi L} \sum_{\substack{\beta+i\zeta \\ \in Z^{(s)}}} \left\{ \frac{1}{(\omega - \beta)^k} - \frac{1}{(\omega - \beta^*)^k} \right\} \quad (\text{B.11})$$

The first summand is bounded by $|\beta - \beta^*|$ while the second is bounded by

$$\left| \frac{1}{(\omega - \beta)^k} - \frac{1}{(\omega - \beta^*)^k} \right| \leq \frac{C \cdot |\beta - \beta^*|}{|\omega - \beta^*|^k \cdot |\omega - \beta|^k}. \quad (\text{B.12})$$

These allow one to conclude relatively to the $W_k^\infty(\mathcal{C})$, $k \geq 1$, bounds on $\widehat{\xi}_{\Upsilon_{\text{sing}}}$. Finally, (B.8) follows from the string centre spacing hypothesis (3.37) and bounds analogous to (B.10) with the root $\beta_a^{(s)}$ removed.

B.2 The functions $\widehat{u}_\Omega^{(\epsilon)}$ on \mathcal{C}

Lemma B.3. *For any $k \in \mathbb{N}$, it holds*

$$\|\widehat{u}_\Gamma^{(\epsilon)} - \widehat{u}_{\Gamma_{\text{reg}}}^{(\epsilon)}\|_{W_k^\infty(\mathcal{C}^{(\epsilon)})} \leq C \cdot L^{k+1} \|\widehat{\xi}_{\Upsilon_{\text{sing}}}\|_{W_k^\infty(\mathcal{C})}. \quad (\text{B.13})$$

Finally, for $\Omega \in \{\Upsilon, \Lambda_b^{(\alpha)}\}$, one has

$$\|\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}\|_{L^1(\mathcal{C}^{(\epsilon)})} = O(L^{-1}) \quad \text{and} \quad \|(\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)})'\|_{L^1(\mathcal{C}^{(\epsilon)})} = O(1) \quad (\text{B.14})$$

as well as

$$\left\| \ln |\widehat{q}_v - *| \cdot \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)} \right\|_{L^1(\mathcal{C}^{(\epsilon)})} = O\left(\frac{\ln L}{L}\right) \quad \text{for } v \in \{L, R\}. \quad (\text{B.15})$$

Proof—

One has

$$\widehat{u}_\Gamma^{(\epsilon)}(s) - \widehat{u}_{\Gamma_{\text{reg}}}^{(\epsilon)}(s) = \int_0^1 \frac{2i\pi\epsilon L \widehat{\xi}_{\Upsilon_{\text{sing}}}(s)}{1 - e^{-2i\pi L \epsilon \widehat{\xi}_{\Upsilon_i}(s)}} dt \quad \text{with} \quad \widehat{\xi}_{\Upsilon_i}(s) = \widehat{\xi}_{\Upsilon_{\text{reg}}}(s) + t \widehat{\xi}_{\Upsilon_{\text{sing}}}(s). \quad (\text{B.16})$$

Then, given some $\delta > 0$ small enough, for $v \in \{L, R\}$, define the contours $\mathcal{C}^{(\text{out})}$, $\mathcal{C}^{(\text{out}; \epsilon)}$, $\mathcal{C}^{(v)}$ and $\mathcal{C}^{(v; \epsilon)}$ as in (A.17)-(A.18). Then, on $\mathcal{C}^{(v; \epsilon)}$ one has the parametrisation $s_x = \widehat{\xi}_\Lambda^{-1}(n_v/L + i\epsilon x)$ with $n_R = |\Lambda| + \tau_R + 1/2$, $n_L = \tau_L + 1/2$ and $x \in [0; \delta]$. This leads to

$$e^{-2i\pi L \epsilon \widehat{\xi}_{\Upsilon_i}(s_x)} - 1 = -\left(e^{2\pi L x} e^{2i\pi \epsilon \widehat{\gamma}_v(s_x)} + 1\right) \quad \text{with} \quad \widehat{\gamma}_v(s) = \widehat{F}_{\text{reg}}(s) - \tau_v - \varkappa_v - t L \widehat{\xi}_{\Upsilon_{\text{sing}}}(s). \quad (\text{B.17})$$

The bounds (3.43), (B.7) and the estimate $\|\widehat{F}_{\text{reg}}\|_{W_1^\infty(\mathcal{S}_\delta(\mathbb{R}))} < Cn_{\text{tot}}^{(\text{msv})}$ then ensure that,

$$\widehat{\gamma}_v(s_x) = \Re(\widehat{\gamma}_v(s_0)) + x(v_v(s_x) + i w_v(s_x)) \quad \text{with} \quad \begin{cases} |\Re(\widehat{\gamma}_v(s_0))| < \frac{1}{2} - \frac{3}{4}\epsilon_r \\ |v_v(s_x)| + |w_v(s_x)| \leq Cn_{\text{tot}}^{(\text{msv})} \end{cases} . \quad (\text{B.18})$$

One has

$$\begin{aligned} \left| e^{-2i\pi L \epsilon \widehat{\xi}_{\Gamma_t}(s_x)} - 1 \right| &\geq e^{2\pi x(L - \epsilon w_v(s_x))} \cdot \left| 1 + e^{-2\pi x(L - \epsilon v_v(s_x))} e^{-2i\pi x v_v(s_x)} \right| \\ &\geq \begin{cases} 1 - \cos(\pi \epsilon_r) & , x \in \left[0; \frac{\epsilon_r}{4Cn_{\text{tot}}^{(\text{msv})}} \right] \\ 1/2 & , x \in \left[\frac{\epsilon_r}{4Cn_{\text{tot}}^{(\text{msv})}}; \delta \right] \end{cases} > c > 0 . \quad (\text{B.19}) \end{aligned}$$

Note that the lower bound on $\left[\frac{\epsilon_r}{4Cn_{\text{tot}}^{(\text{msv})}}; \delta \right]$ follows from $e^{-2\pi x(L - \epsilon v_{R/L}(s_x))} < 1/2$ for L large enough.

Thus, the above lower bound and (B.16) imply that, for $s \in \mathcal{C}^{(v;\epsilon)}$, one has

$$\left| \widehat{u}_{\Gamma}^{(\epsilon)}(s) - \widehat{u}_{\Gamma_{\text{reg}}}^{(\epsilon)}(s) \right| \leq C \left| \widehat{\xi}_{\Gamma_{\text{sing}}}(s) \right| . \quad (\text{B.20})$$

Next, for $s \in \mathcal{C}^{(\text{out};\epsilon)}$, one has the direct bounds

$$\left| \widehat{F}_{\text{reg}}(s) + tL \widehat{\xi}_{\Gamma_{\text{sing}}}(s) \right| \leq C \quad \text{and} \quad \epsilon \Im(\widehat{\xi}_{\Lambda}(s)) > \delta \quad (\text{B.21})$$

so that, for L large enough,

$$\left| e^{-2i\pi L \epsilon \widehat{\xi}_{\Gamma_t}(s_x)} - 1 \right| \geq e^{2\pi(L\delta - C)} - 1 > 1 . \quad (\text{B.22})$$

Hence, the bounds (B.20) also holds for $s \in \mathcal{C}^{(\text{out};\epsilon)}$. The above entails (B.13) for $k = 0$. The result for general k can be obtained by taking derivatives of (B.16) and using similar types of bounds as described above.

It remains to establish the last set of bounds (B.14)-(B.15). By using the contours introduced in (A.17)-(A.18) and the fact that $C > |(\widehat{q}_v - s)/(\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(s))| > C^{-1} > 0$ on $\mathcal{C}^{(v)}$, and that $|\ln |\widehat{q}_v - s||$ is bounded on $\mathcal{C} \setminus \mathcal{C}^{(v)}$ one gets

$$\begin{aligned} \int_{\mathcal{C}^{(\epsilon)}} (1 + |\ln |\widehat{q}_v - s||) \left| \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \right| \cdot |ds| &\leq C_1 \int_{\mathcal{C}^{(\text{out};\epsilon)}} \left| \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \right| \cdot |ds| + C_2 \int_{\mathcal{C}^{(\overline{v};\epsilon)}} \left| \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \right| \cdot |ds| \\ &+ \int_{\mathcal{C}^{(v;\epsilon)}} (C_3 + |\ln |\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(s)||) \left| \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \right| \cdot |ds| . \quad (\text{B.23}) \end{aligned}$$

Here, $\overline{v} = L$ if $v = R$ and *vice versa*. Since $|e^{2i\pi \epsilon L \widehat{\xi}_{\Omega_{\text{reg}}}}| \leq C e^{-c' L}$ with $c' > 0$ on $\mathcal{C}^{(\text{out};\epsilon)}$, the first term will only generate exponentially small corrections. The second term is similar to the third one, so that it remains to focus on the last line. There, implementing the change of variable $s_{\epsilon t}$ with s_t as defined in (A.19), leads to

$$\int_{\mathcal{C}^{(v;\epsilon)}} (1 + |\ln |\widehat{\xi}_{\Lambda}(\widehat{q}_v) - \widehat{\xi}_{\Lambda}(s)||) \left| \widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)}(s) \right| \cdot |ds| \leq \int_0^{2\pi L \delta} \left(C + \left| \ln \left| \frac{t}{2\pi L} \right| \right| \right) \ln [1 + e^{-|t| + 2i\pi \epsilon \sigma_v s_v^{(\Omega)}(s_{\epsilon t})}] \cdot \frac{dt}{2\pi L} \quad (\text{B.24})$$

with $\zeta_v^{(\Omega)}$ as defined in (A.6). It then remains to observe that bounds similar to (??) allow one to use the bounds $|\ln(1+z)| \leq C|z|$ for $\arg(z) \in]\eta - \pi; \pi - \eta[$ with $\eta > 0$ and fixed. The latter bound entails the claim. Finally, the bounds relative to $(\widehat{u}_{\Omega_{\text{reg}}}^{(\epsilon)})'$ can be obtained through similar handlings. ■

C Auxiliary integrals and special functions

C.1 A few auxiliary integrals

Lemma C.1. *Let $a_{\pm} \in \mathbb{H}_{\pm}$ and $\Im(\alpha) \in]-\pi/2; \pi/2[$, then*

$$\int_{\mathbb{R}} \frac{\ln(1 + e^{-|t| + \alpha \text{sgn}(t)})}{t - a_{\pm}} \frac{dt}{2i\pi} = \mp \ln \Gamma\left(\frac{1}{2} \pm \frac{a_{\pm} - \alpha}{2i\pi}\right) \pm \frac{1}{2} \ln(2\pi) + \frac{a_{\pm} - \alpha}{2i\pi} \ln\left(\frac{a_{\pm}}{\pm 2i\pi}\right) - \frac{a_{\pm}}{2i\pi}. \quad (\text{C.1})$$

Proof— We set $f(t) = (t - a_{\pm}) \ln(t - a_{\pm}) - (t - a_{\pm})$ and integrate twice by parts

$$\int_{\mathbb{R}} \frac{\ln(1 + e^{-|t| + \alpha \text{sgn}(t)})}{t - a_{\pm}} \frac{dt}{2i\pi} = \int_{\mathbb{R}} \frac{f(t)}{(e^{\frac{t-\alpha}{2}} + e^{-\frac{t-\alpha}{2}})^2} \frac{dt}{2i\pi} - \frac{f(0)}{2i\pi} - \alpha \frac{f'(0)}{2i\pi}. \quad (\text{C.2})$$

As f is holomorphic in the lower/upper half plane (depending whether a_{\pm} belongs to the upper/lower half plane), one can deform the integration contour to $\mathbb{R} \mp 2i\pi N$, $N \in \mathbb{N}$, for the price of picking the poles at $t = \mp i\pi(2p+1)$, with $p = 1, \dots, N-1$. Then

$$\int_{\mathbb{R}} \frac{\ln(1 + e^{-|t| + \alpha \text{sgn}(t)})}{t - a_{\pm}} \frac{dt}{2i\pi} = \pm \sum_{p=0}^{N-1} f'(\mp i\pi(2p+1) + \alpha) + \int_{\mathbb{R} \mp 2i\pi N} \frac{f(t)}{(e^{\frac{t-\alpha}{2}} + e^{-\frac{t-\alpha}{2}})^2} \frac{dt}{2i\pi} - \frac{f(0)}{2i\pi} - \alpha \frac{f'(0)}{2i\pi}. \quad (\text{C.3})$$

Now, one integrates twice by parts and compute the sum over the crossed poles in terms of Γ functions

$$\begin{aligned} \int_{\mathbb{R}} \frac{\ln(1 + e^{-|t| + \alpha \text{sgn}(t)})}{t - a_{\pm}} \frac{dt}{2i\pi} &= \frac{f(\mp 2i\pi N) - f(0)}{2i\pi} + \alpha \frac{f'(\mp 2i\pi N) - f'(0)}{2i\pi} \pm N \ln(\mp 2i\pi) \\ &\quad \pm \ln \Gamma\left(N + \frac{1}{2} \pm \frac{a_{\pm} - \alpha}{2i\pi}\right) \mp \ln \Gamma\left(\frac{1}{2} \pm \frac{a_{\pm} - \alpha}{2i\pi}\right) + \int_{\mathbb{R}} \frac{\ln(1 + e^{-|t| + \alpha \text{sgn}(t)})}{\mp 2i\pi N + t - a_{\pm}} \frac{dt}{2i\pi}. \end{aligned} \quad (\text{C.4})$$

The last integral is a $O(N^{-1})$ by the dominated convergence theorem and

$$\frac{f(\mp 2i\pi N)}{2i\pi} + \alpha \frac{f'(\mp 2i\pi N)}{2i\pi} \pm N \ln(\mp 2i\pi) \pm \ln \Gamma\left(N + \frac{1}{2} \pm \frac{a_{\pm} - \alpha}{2i\pi}\right) = \pm \frac{1}{2} \ln 2\pi + \frac{\alpha - a_{\pm}}{2i\pi} \ln(\mp 2i\pi) + O\left(\frac{1}{N}\right). \quad (\text{C.5})$$

Putting all these results together and then using that

$$-\frac{a_{\pm} - \alpha}{2i\pi} \ln\left(\frac{\mp 1}{2i\pi}\right) + \frac{a_{\pm} - \alpha}{2i\pi} \ln(-a_{\pm}) = \frac{a_{\pm} - \alpha}{2i\pi} \ln\left(\frac{a_{\pm}}{\pm 2i\pi}\right) \quad (\text{C.6})$$

yields the value of the integral. ■

Lemma C.2. *Let $a, b \in \mathbb{C}$ be small enough, then*

$$\int_{\mathbb{R}} \left\{ \frac{\text{sgn}(t)}{1 + e^{|t|-a\text{sgn}(t)}} - \frac{\text{sgn}(t)}{1 + e^{|t|-(a+b)\text{sgn}(t)}} \right\} \ln \Gamma \left(\frac{\frac{1}{2} + \frac{t-a}{2i\pi}}{\frac{1}{2} + \frac{t-a-b}{2i\pi}} \right) \cdot \frac{dt}{2i\pi} = -\ln G \left(1 - \frac{b}{2i\pi}, 1 + \frac{b}{2i\pi} \right) \quad (\text{C.7})$$

and

$$\int_{\mathbb{R}} \left\{ \frac{\text{sgn}(t)}{1 + e^{|t|-a\text{sgn}(t)}} - \frac{\text{sgn}(t)}{1 + e^{|t|-(a+b)\text{sgn}(t)}} \right\} \cdot \frac{dt}{2i\pi} = -\frac{b}{2i\pi}. \quad (\text{C.8})$$

Here, we agree upon $G(x, y) = G(x)G(y)$ and upon similar notations for products of Γ functions. Also, one has that

$$\int_{\mathbb{R}} \left\{ \frac{\text{sgn}(t)}{1 + e^{|t|-a\text{sgn}(t)}} - \frac{\text{sgn}(t)}{1 + e^{|t|-(a+b)\text{sgn}(t)}} \right\} \ln [-i(t + i0^+)] \cdot \frac{dt}{2i\pi} = \ln \Gamma \left(\frac{\frac{1}{2} + \frac{a}{2i\pi}}{\frac{1}{2} + \frac{a+b}{2i\pi}} \right) - \frac{b}{2i\pi} \ln(2\pi). \quad (\text{C.9})$$

Proof—

The integrals can be computed either by means of direct integration or owing to a residue calculation in the spirit of Lemma C.1 after observing that

$$\frac{\text{sgn}(t)}{1 + e^{|t|-a\text{sgn}(t)}} - \frac{\text{sgn}(t)}{1 + e^{|t|-(a+b)\text{sgn}(t)}} = \frac{1}{1 + e^{t-a}} - \frac{1}{1 + e^{t-(a+b)}}. \quad (\text{C.10})$$

■

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